MATH 415

Modern Algebra I

Lecture 16:
Some examples of rings.
Field of quotients.

Rings

Definition. A **ring** is a set R, together with two binary operations usually called **addition** and **multiplication** and denoted accordingly, such that

- R is an abelian group under addition,
- R is a semigroup under multiplication,
- multiplication distributes over addition.

The complete list of axioms is as follows:

(A0) for all $x, y \in R$, $x + y$ is an element of R;
(A1) $(x + y) + z = x + (y + z)$ for all $x, y, z \in R$;
(A2) there exists an element, denoted 0, in R such that $x + 0 = 0 + x = x$ for all $x \in R$;
(A3) for every $x \in R$ there exists an element, denoted $-x$, in R such that $x + (-x) = (-x) + x = 0$;
(A4) $x + y = y + x$ for all $x, y \in R$;
(M0) for all $x, y \in R$, xy is an element of R;
(M1) $(xy)z = x(yz)$ for all $x, y, z \in R$;
(D) $x(y+z) = xy+xz$ and $(y+z)x = yx+zx$ for all $x, y, z \in R$.

Ring of functions

Let R be a ring and S be a nonempty set. Denote by $\mathcal{F}(S, R)$ the set of all functions $f : S \to R$. Given $f, g \in \mathcal{F}(S, R)$, we let $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for all $x \in S$. That is, to add (resp. multiply) functions, we add (resp. multiply) their values at every point. Then $\mathcal{F}(S, R)$ is a ring.

The ring $\mathcal{F}(S, R)$ inherits many properties from the ring R, with one important exception. If R is a nontrivial ring and S has more than one element, then the ring $\mathcal{F}(S, R)$ has divisors of zero. Indeed, take any point $x_0 \in S$, any nonzero element $r \in R$, and let

$$f_1(x) = \begin{cases} r & \text{if } x = x_0, \\ 0 & \text{if } x \in S \setminus \{x_0\}; \end{cases} \quad f_2(x) = \begin{cases} 0 & \text{if } x = x_0, \\ r & \text{if } x \in S \setminus \{x_0\}. \end{cases}$$

Then the functions f_1 and f_2 are nonzero elements of the ring $\mathcal{F}(S, R)$ while $f_1 f_2 = 0$.

Ring of matrices

Let R be a ring. For any integers $m, n > 0$, denote by $\mathcal{M}_{m,n}(R)$ the set of all $m\times n$ matrices with entries from R. Given two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathcal{M}_{m,n}(R)$, we let $A + B = (c_{ij})$ and $A - B = (d_{ij})$, where $c_{ij} = a_{ij} + b_{ij}$ and $d_{ij} = a_{ij} - b_{ij}$, $1 \le i \le m$, $1 \le j \le n$. Given matrices $A = (a_{ij}) \in \mathcal{M}_{m,n}(R)$ and $B = (b_{ij}) \in \mathcal{M}_{n,p}(R)$, we let $AB = (c_{ij})$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$, $1 \le i \le m$, $1 \le j \le p$.

Matrix multiplication is associative. Indeed, let $A = (a_{ij}) \in \mathcal{M}_{m,n}(R)$, $B = (b_{jk}) \in \mathcal{M}_{n,p}(R)$ and $C = (c_{k\ell}) \in \mathcal{M}_{p,q}(R)$. Then $(AB)C = (d_{i\ell})$ and $A(BC) = (d'_{i\ell})$ are matrices in $\mathcal{M}_{n,q}(R)$. Using distributive laws in R, we obtain that

$$d_{i\ell} = \sum_{k=1}^{p} \sum_{j=1}^{n} (a_{ij}b_{jk})c_{k\ell}, \quad d'_{i\ell} = \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij}(b_{jk}c_{k\ell}).$$

Hence $(AB)C = A(BC)$ since R is a ring.

As a consequence, square matrices in $\mathcal{M}_{n,n}(R)$ form a ring.

Direct product of rings

Suppose $R_1, R_2, \ldots, R_n$ are rings. We define addition and multiplication on the Cartesian product $R_1 \times R_2 \times \cdots \times R_n$ by

$$(r_1, r_2, \ldots, r_n) + (r_1', r_2', \ldots, r_n') = (r_1 + r_1', r_2 + r_2', \ldots, r_n + r_n'),$$
$$(r_1, r_2, \ldots, r_n)(r_1', r_2', \ldots, r_n') = (r_1 r_1', r_2 r_2', \ldots, r_n r_n')$$

for all $r_i, r_i' \in R_i$, $1 \leq i \leq n$.

Then $R_1 \times R_2 \times \cdots \times R_n$ is a ring called the **direct product** of rings $R_1, R_2, \ldots, R_n$.

The ring $R_1 \times R_2 \times \cdots \times R_n$ is commutative if each of the rings $R_1, R_2, \ldots, R_n$ is commutative. It is a ring with unity if each of the rings $R_1, R_2, \ldots, R_n$ has the unity.

If at least two of the rings $R_1, R_2, \ldots, R_n$ are nontrivial, then the direct product $R_1 \times R_2 \times \cdots \times R_n$ admits divisors of zero.

Complex numbers

$\mathbb{C}$: complex numbers.

Complex number: $\boxed{z = x + iy,}$

where $x, y \in \mathbb{R}$ and $i^2 = -1$.

$i = \sqrt{-1}$: imaginary unit

Alternative notation: $z = x + yi$.

$x =$ real part of z,
$iy =$ imaginary part of z

$y = 0 \implies z = x$ (real number)
$x = 0 \implies z = iy$ (purely imaginary number)

We add, subtract, and multiply complex numbers as polynomials in i (but keep in mind that $i^2 = -1$).

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2),$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Given $z = x + iy$, the **complex conjugate** of z is $\bar{z} = x - iy$. The **modulus** of z is $|z| = \sqrt{x^2 + y^2}$.

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2.$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}, \qquad (x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

Complex exponentials

Definition. For any $z \in \mathbb{C}$ let

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

Remark. A sequence of complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2, \ldots$ converges to $z = x + iy$ if $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Theorem 1 If $z = x + iy,\ x, y \in \mathbb{R},$ then

$$e^z = e^x(\cos y + i \sin y).$$

In particular, $e^{i\phi} = \cos\phi + i\sin\phi,\ \phi \in \mathbb{R}.$

Theorem 2 $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}.$

Proposition $e^{i\phi} = \cos\phi + i\sin\phi$ for all $\phi \in \mathbb{R}$.

Proof: $e^{i\phi} = 1 + i\phi + \dfrac{(i\phi)^2}{2!} + \cdots + \dfrac{(i\phi)^n}{n!} + \cdots$

The sequence $1, i, i^2, i^3, \ldots, i^n, \ldots$ is periodic:
$\underbrace{1, i, -1, -i}, \underbrace{1, i, -1, -i}, \ldots$

It follows that

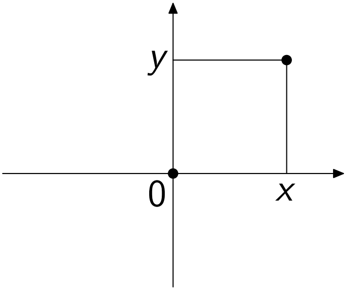
$$e^{i\phi} = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots + (-1)^k \frac{\phi^{2k}}{(2k)!} + \cdots$$

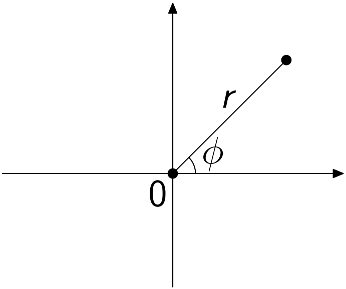
$$+\ i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots + (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} + \cdots\right)$$

$$= \cos\phi + i\sin\phi.$$

Geometric representation

Any complex number $z = x + iy$ is represented by the vector/point $(x, y) \in \mathbb{R}^2$.





$$x = r\cos\phi, \; y = r\sin\phi \implies z = r(\cos\phi + i\sin\phi) = re^{i\phi}$$

If $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\phi_1+\phi_2)}$, $z_1/z_2 = (r_1/r_2)e^{i(\phi_1-\phi_2)}$.

Complex numbers as an $\mathbb{R}$-algebra

Complex numbers can be defined as a certain 2-dimensional algebra over the field $\mathbb{R}$. We have a distinguished basis $\mathbf{1}, i$. Hence every complex number z is uniquely represented as $z = x\mathbf{1} + yi$, where $x, y \in \mathbb{R}$.

Since multiplication is a bilinear function, it is enough to define $z_1 \cdot z_2$ in the case $z_1, z_2 \in \{\mathbf{1}, i\}$. We set $\mathbf{1} \cdot \mathbf{1} = \mathbf{1}$, $\mathbf{1} \cdot i = i \cdot \mathbf{1} = i$ and $i \cdot i = -\mathbf{1}$.

Because of bilinearity of the product, it is easy to check that $\mathbf{1} \cdot z = z \cdot \mathbf{1}$, $z_1 \cdot z_2 = z_2 \cdot z_1$ and $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$.

Quaternions

The **Hamilton quaternions** $\mathbb{H}$ can be defined as a certain 4-dimensional algebra over the field $\mathbb{R}$. We have a distinguished basis $\mathbf{1}, i, j, k$. Hence every quaternion q is uniquely represented as $z = a\mathbf{1} + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$.

Since multiplication is a bilinear function, it is enough to define $q_1 \cdot q_2$ for $q_1, q_2 \in \{\mathbf{1}, i, j, k\}$. We set $\mathbf{1} \cdot \mathbf{1} = \mathbf{1}$, $\mathbf{1} \cdot i = i \cdot \mathbf{1} = i$, $\mathbf{1} \cdot j = j \cdot \mathbf{1} = j$, $\mathbf{1} \cdot k = k \cdot \mathbf{1} = k$, $i \cdot i = j \cdot j = k \cdot k = -\mathbf{1}$, $i \cdot j = k$, $j \cdot i = -k$, $j \cdot k = i$, $k \cdot j = -i$, $k \cdot i = j$, $i \cdot k = -j$.

Theorem $\mathbb{H}$ is a non-commutative division ring.

Lemma 1 $q \cdot \mathbf{1} = \mathbf{1} \cdot q = q$ for all $q \in \mathbb{H}$.

Proof. Since $f_1(q) = q \cdot \mathbf{1}$, $f_2(q) = \mathbf{1} \cdot q$ and $f_3(q) = q$ are all linear functions on $\mathbb{H}$, it is enough to prove the equalities in the case when $q \in \{\mathbf{1}, i, j, k\}$. In this case they follow from the definition of multiplication.

Lemma 2 For any $a, b \in \mathbb{R}$ and $q \in \mathbb{H}$ we have $(a\mathbf{1}) + (b\mathbf{1}) = (a+b)\mathbf{1}$, $(a\mathbf{1}) \cdot (b\mathbf{1}) = (ab)\mathbf{1}$ and $(a\mathbf{1}) \cdot q = aq$.

In view of Lemma 2, we can identify any quaternion of the form $a\mathbf{1}$ with the real number a so that $\mathbb{R} \subset \mathbb{H}$. This also allows to consider scalar multiplication on $\mathbb{H}$ as a special case of multiplication of quaternions. In particular, we can use the same notation $q_1 q_2$ for both kinds of multiplication.

Lemma 3 Multiplication of quaternions is associative.

Idea of the proof. Since $(q_1q_2)q_3$ and $q_1(q_2q_3)$ are both trilinear functions of $q_1, q_2, q_3 \in \mathbb{H}$, it is enough to prove the equality $(q_1q_2)q_3 = q_1(q_2q_3)$ in the case when $q_1, q_2, q_3 \in \{\mathbf{1}, i, j, k\}$.

For any quaternion $q = a + bi + cj + dk$, we define the **conjugate** quaternion by $\bar{q} = a - bi - cj - dk$ and the **modulus** of q by $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Lemma 4 $q\bar{q} = \bar{q}q = |q|^2$ for all $q \in \mathbb{H}$.

Lemma 5 Every nonzero quaternion q has a multiplicative inverse: $q^{-1} = |q|^{-2}\bar{q}$.

Rational quaternions are quaternions of the form $q = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{Q}$. The rational quaternions also form a division ring.

Integer quaternions are quaternions of the form $q = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{Z}$. The integer quaternions form a ring. This ring has only 8 invertible elements (the units): $\pm 1, \pm i, \pm j, \pm k$. These 8 elements form a group under quaternion multiplication, called **the quaternion group** and denoted Q_8.

Theorem Any non-abelian group of order 8 is isomorphic either to the dihedral group D_4 or to the quaternion group Q_8.

From a ring to a field

Question 1. When a ring R can be extended to a field?

An obvious necessary condition is commutativity. Another necessary condition is absence of zero divisors (which is equivalent to cancellation laws).

Proposition If an element of a ring with unity has a multiplicative inverse, then it is not a divisor of zero.

Question 2. When a semigroup S can be extended to a group?

Theorem Any finite semigroup with cancellation is a group.

Theorem If S is a commutative semigroup with cancellation, then it can be extended to an abelian group G. Moreover, if $G = \langle S \rangle$, then any element of G is of the form $b^{-1}a$, where $a, b \in S$. Moreover, if $G = \langle S \rangle$, then the group G is unique up to isomorphism.

Suppose S is a commutative semigroup with cancellation (with multiplicative notation). Consider the direct product $S \times S$. It is also a commutative semigroup with cancellation. For any $(a, b) \in S \times S$ we are going to use an alternative notation $\frac{a}{b}$.

Then the operation on $S \times S$ is given by $\frac{a}{b} \cdot \frac{a'}{b'} = \frac{aa'}{bb'}$.

Let $\sim$ be a relation on $S \times S$ such that

$$\frac{a_1}{b_1} \sim \frac{a_2}{b_2} \text{ if and only if } a_1 b_2 = a_2 b_1.$$

Lemma 1 $\sim$ is an equivalence relation.

Proof. Reflexivity and symmetry are obvious. To prove transitivity, assume $\frac{a_1}{b_1} \sim \frac{a_2}{b_2}$ and $\frac{a_2}{b_2} \sim \frac{a_3}{b_3}$, that is, $a_1 b_2 = a_2 b_1$ and $a_2 b_3 = a_3 b_2$. Using commutativity in S, we obtain $a_1 b_3 b_2 = a_1 b_2 b_3 = a_2 b_1 b_3 = a_2 b_3 b_1 = a_3 b_2 b_1 = a_3 b_1 b_2$. After cancellation, $a_1 b_3 = a_3 b_1$, that is, $\frac{a_1}{b_1} \sim \frac{a_3}{b_3}$.

Lemma 2 The relation $\sim$ is compatible with the operation on $S \times S$.

Consider the factor space $G = (S \times S)/\sim$ with the operation induced by the operation on $S \times S$. It is a commutative semigroup. For any $a, b \in S$ we denote the equivalence class of $\frac{a}{b}$ by $\left[\frac{a}{b}\right]$.

Lemma 3 $\left[\frac{ac}{bc}\right] = \left[\frac{a}{b}\right]$ for all $a, b, c \in S$.

Lemma 4 $\left[\frac{c}{c}\right]$ is an identity element in G for any $c \in S$.

Lemma 5 $\left[\frac{b}{a}\right] = \left[\frac{a}{b}\right]^{-1}$ for all $a, b \in S$.

Lemma 6 G is an abelian group.

Lemma 7 Let $c \in S$. The map $f : S \to G$ defined by $f(a) = \left[\frac{ac}{c}\right]$ is an injective homomorphism.

Lemma 8 $\left[\frac{a}{b}\right] = (f(b))^{-1} f(a)$ for all $a, b \in S$.

Field of quotients

Theorem A ring R with unity can be extended to a field if and only if it is an integral domain.

If R is an integral domain, then there is a (smallest) field F containing R called the **quotient field** of R (or the **field of quotients**). Any element of F is of the form $b^{-1}a$, where $a, b \in R$. The field F is unique up to isomorphism.

Examples. • The quotient field of $\mathbb{Z}$ is $\mathbb{Q}$.
• The quotient field of $\mathbb{R}[X]$ is $\mathbb{R}(X)$.
• The quotient field of $\mathbb{Z}[\sqrt{2}] = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$ is $\mathbb{Q}[\sqrt{2}] = \{p + q\sqrt{2} \mid p, q \in \mathbb{Q}\}$.