Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Find a quadratic polynomial $p(x)$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

Let $p(x) = a + bx + cx^2$. Then $p(-1) = a - b + c$, $p(1) = a + b + c$, and $p(3) = a + 3b + 9c$. Also, $p'(x) = b + 2cx$ so that $p'(2) = b + 4c$. The coefficients a, b, and c are to be chosen so that

$$\begin{cases} a - b + c = 6, \\ a + 3b + 9c = 6, \\ b + 4c = a + b + c \end{cases} \iff \begin{cases} a - b + c = 6, \\ a + 3b + 9c = 6, \\ a - 3c = 0. \end{cases}$$

This is a system of linear equations. To solve it, we convert the augmented matrix to reduced row echelon form using elementary row operations:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 1 & 3 & 9 & 6 \\ 1 & 0 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 1 & -1 & 1 & 6 \\ 1 & 3 & 9 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & -1 & 4 & 6 \\ 1 & 3 & 9 & 6 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & -1 & 4 & 6 \\ 0 & 3 & 12 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & -1 & 4 & 6 \\ 0 & 0 & 24 & 24 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & -1 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

We obtain that the system has a unique solution: $a = 3$, $b = -2$, and $c = 1$. Thus $p(x) = x^2 - 2x + 3$.

Problem 2 (20 pts.) Consider a linear transformation $L : \mathbb{R}^5 \to \mathbb{R}^2$ given by

$$L(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_3 + x_5,\ 2x_1 - x_2 + x_4).$$

Find a basis for the null-space of L, then extend it to a basis for $\mathbb{R}^5$.

The null-space $\mathcal{N}(L)$ consists of all vectors $\mathbf{x} \in \mathbb{R}^5$ such that $L(\mathbf{x}) = \mathbf{0}$. This is the solution set of the following systems of linear equations:

$$\begin{cases} x_1 + x_3 + x_5 = 0 \\ 2x_1 - x_2 + x_4 = 0 \end{cases} \iff \begin{cases} x_1 + x_3 + x_5 = 0 \\ -x_2 - 2x_3 + x_4 - 2x_5 = 0 \end{cases}$$

$$\iff \begin{cases} x_1 + x_3 + x_5 = 0 \\ x_2 + 2x_3 - x_4 + 2x_5 = 0 \end{cases} \iff \begin{cases} x_1 = -x_3 - x_5 \\ x_2 = -2x_3 + x_4 - 2x_5 \end{cases}$$

The general solution of the system is

$$\mathbf{x} = (-t_1 - t_3, -2t_1 + t_2 - 2t_3, t_1, t_2, t_3) = t_1(-1, -2, 1, 0, 0) + t_2(0, 1, 0, 1, 0) + t_3(-1, -2, 0, 0, 1),$$

where t_1, t_2, t_3 are arbitrary real numbers. We obtain that the null-space $\mathcal{N}(L)$ is spanned by vectors $\mathbf{v}_1 = (-1,-2,1,0,0)$, $\mathbf{v}_2 = (0,1,0,1,0)$, and $\mathbf{v}_3 = (-1,-2,0,0,1)$. The last three coordinates of these vectors form the standard basis for $\mathbb{R}^3$. It follows that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. Hence they form a basis for $\mathcal{N}(L)$.

To extend the basis for $\mathcal{N}(L)$ to a basis for $\mathbb{R}^5$, we need two more vectors. We can use two vectors from the standard basis. For example, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_1, \mathbf{e}_2$ form a basis for $\mathbb{R}^5$. To verify this, we show that a 5×5 matrix with these vectors as columns has a nonzero determinant:

$$\begin{vmatrix} -1 & 0 & -1 & 1 & 0 \\ -2 & 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & -2 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1.$$

Problem 3 (20 pts.) Let $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,1,0)$, and $\mathbf{v}_3 = (1,0,1)$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $T(\mathbf{v}_1) = \mathbf{v}_2$, $T(\mathbf{v}_2) = \mathbf{v}_3$, $T(\mathbf{v}_3) = \mathbf{v}_1$. Find the matrix of the operator T relative to the standard basis.

Let U be a 3×3 matrix such that its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To determine whether $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$, we find the determinant of U:

$$\det U = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Since $\det U \neq 0$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. Therefore they form a basis for $\mathbb{R}^3$. It follows that the operator T is defined well and uniquely.

The matrix of the operator T relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since the matrix U is the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis, the matrix of T relative to the standard basis is $A = UBU^{-1}$.

To find the inverse U^{-1}, we merge the matrix U with the identity matrix I into one 3×6 matrix and apply row reduction to convert the left half U of this matrix into I. Simultaneously, the right half I will be converted into U^{-1}:

$$(U|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) = (I|U^{-1}).$$

Thus

$$A = UBU^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

Problem 4 (20 pts.) Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be the operator of orthogonal reflection in the plane Π spanned by vectors $\mathbf{u}_1 = (1, 0, -1)$ and $\mathbf{u}_2 = (1, -1, 3)$. Find the image of the vector $\mathbf{u} = (2, 3, 4)$ under this operator.

By definition of the orthogonal reflection, $R(\mathbf{x}) = \mathbf{x}$ for any vector $\mathbf{x} \in \Pi$ and $R(\mathbf{y}) = -\mathbf{y}$ for any vector $\mathbf{y}$ orthogonal to the plane Π. The vector $\mathbf{u}$ is uniquely decomposed as $\mathbf{u} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in \Pi$ and $\mathbf{o} \in \Pi^\perp$. Then $R(\mathbf{u}) = R(\mathbf{p} + \mathbf{o}) = R(\mathbf{p}) + R(\mathbf{o}) = \mathbf{p} - \mathbf{o}$.

The component $\mathbf{p}$ is the orthogonal projection of the vector $\mathbf{u}$ onto the plane Π. We can compute it using the formula

$$\mathbf{p} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2,$$

in which $\mathbf{v}_1, \mathbf{v}_2$ is an arbitrary orthogonal basis for Π. To get such a basis, we apply the Gram-Schmidt process to the basis $\mathbf{u}_1, \mathbf{u}_2$:

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 = (1, 0, -1), \\ \mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = (1, -1, 3) - \frac{-2}{2}(1, 0, -1) = (2, -1, 2). \end{aligned}$$

Now

$$\mathbf{p} = \frac{-2}{2}(1, 0, -1) + \frac{9}{9}(2, -1, 2) = (1, -1, 3).$$

Then $\mathbf{o} = \mathbf{u} - \mathbf{p} = (1, 4, 1)$. Finally, $R(\mathbf{u}) = \mathbf{p} - \mathbf{o} = (0, -5, 2)$.

Problem 5 (25 pts.) Consider the vector space W of all polynomials of degree at most 3 in variables x and y with real coefficients. Let D be a linear operator on W given by $D(p) = \dfrac{\partial p}{\partial x}$ for any $p \in W$. Find the Jordan canonical form of the operator D.

The vector space W is 10-dimensional. It has a basis of monomials: $1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3$.

Note that $D(x^m y^k) = m x^{m-1} y^k$ if $m > 0$ and $D(x^m y^k) = 0$ otherwise. It follows that the operator D^4 maps each monomial to zero, which implies that this operator is identically zero. As a consequence, 0 is the only eigenvalue of the operator D.

To determine the Jordan canonical form of D, we need to determine the null-spaces of its iterations. Indeed, $\dim \mathcal{N}(D)$ is the total number of Jordan blocks in the Jordan canonical form of D. Next, $\dim \mathcal{N}(D^2) - \dim \mathcal{N}(D)$ is the number of Jordan blocks of dimensions at least 2×2. Further, $\dim \mathcal{N}(D^3) - \dim \mathcal{N}(D^2)$ is the number of Jordan blocks of dimensions at least 3×3, and so on...

The null-space $\mathcal{N}(D)$ is 4-dimensional, it is spanned by $1, y, y^2, y^3$. The null-space $\mathcal{N}(D^2)$ is 7-dimensional, it is spanned by $1, y, y^2, y^3, x, xy, xy^2$. The null-space $\mathcal{N}(D^3)$ is 9-dimensional, it is spanned by $1, y, y^2, y^3, x, xy, xy^2, x^2, x^2y$. The null-space $\mathcal{N}(D^4)$ is the entire 10-dimensional space W. It follows that the Jordan canonical form of D contains one Jordan block of dimensions 1×1, 2×2, 3×3, and 4×4:

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Bonus Problem 6 (15 pts.) An upper triangular matrix is called unipotent if all diagonal entries are equal to 1. Prove that the inverse of a unipotent matrix is also unipotent.

Let $\mathcal{U}$ denote the class of elementary row operations that add a scalar multiple of row $\#i$ to row $\#j$, where i and j satisfy $j < i$. It is easy to see that such an operation transforms a unipotent matrix into another unipotent matrix.

It remains to observe that any unipotent matrix A (which is in row echelon form) can be converted into the identity matrix I (which is its reduced row echelon form) by applying only operations from the class $\mathcal{U}$. Now the same sequence of elementary row operations converts I into the inverse matrix A^{-1}. Since the identity matrix is unipotent, so is A^{-1}.