MATH 423–500/200 May 7, 2012

Final exam (with solutions)

Problem 1 (15 pts.) Find a quadratic polynomial $p(x)$ such that $p(1) = 2$, $p(2) = 5$, and $p(3) = 2p(-2)$.

Solution: $p(x) = x^2 + 1$.

Problem 2 (20 pts.) Let V and W be subspaces of the vector space $\mathbb{R}^n$ such that V is a proper subset of W, i.e., $V \subset W$ and $V \neq W$. Prove that $\dim V < \dim W$.

Any linearly independent set in a vector space can be extended to a basis. Since the vector space $\mathbb{R}^n$ is finite-dimensional, it does not admit infinitely many linearly independent vectors. Clearly, the same is true for the subspaces V and W. It follows that V and W are also finite-dimensional.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be a basis for V. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent in W since they are linearly independent in V. Therefore we can extend this collection of vectors to a basis for W by adding some vectors $\mathbf{w}_1, \dots, \mathbf{w}_m$. As $V \neq W$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ alone do not span W. Hence we do need to add some vectors, i.e., $m \geq 1$. Thus $\dim V = k$ and $\dim W = k + m > k$.

Problem 3 (20 pts.) The vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (1, 2, 1)$ form a basis for $\mathbb{R}^3$. The vectors $\mathbf{w}_1 = (1, 1, 0)$, $\mathbf{w}_2 = (0, 1, 1)$, and $\mathbf{w}_3 = (1, 1, 1)$ form another basis for $\mathbb{R}^3$. Find the transition matrix that changes coordinates from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

Solution: $$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

Problem 4 (20 pts.) Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 1, 1)$, $\mathbf{x}_2 = (-1, 1, 2, 2)$, and $\mathbf{x}_3 = (-3, 1, 5, 1)$.

(i) Find the orthogonal projection of the vector $\mathbf{y} = (0, 0, 24, 0)$ onto the subspace V.

(ii) Find the distance from $\mathbf{y}$ to the subspace V.

Solution: Orthogonal projection: $\mathbf{p} = (-2, 6, 22, -2)$. Distance from $\mathbf{y}$ to V: $\|\mathbf{y} - \mathbf{p}\| = 4\sqrt{3}$.

Problem 5 (25 pts.) Let $A = \begin{pmatrix} 2 & 0 & -2 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$.

(i) Determine whether the matrix A is diagonalizable.

(ii) If A is diagonalizable, find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A. If A is not diagonalizable, find the Jordan canonical form of A.

Solution: A is diagonalizable. Basis of eigenvectors: $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (2, 0, 1)$, $\mathbf{v}_3 = (0, 1, 0)$.

Problem 5′ (25 pts.) Let $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$.

(i) Determine whether the matrix A is diagonalizable.

(ii) If A is diagonalizable, find a basis for $\mathbb{R}^4$ consisting of eigenvectors of A. If A is not diagonalizable, find the Jordan canonical form of A.

Solution: A is not diagonalizable. The Jordan canonical form: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Bonus Problem 6′ (15 pts.) (i) Prove that every normal matrix B can be represented as a product $B = UR$, where the matrix R is Hermitian and the matrix U is unitary.

First we consider the case when B is diagonal, $B = \operatorname{diag}(z_1, z_2, \dots, z_n)$. Any of the complex numbers z_k can be represented as a product $z_k = r_k u_k$, where r_k is real and $|u_k| = 1$. We let $R_1 = \operatorname{diag}(r_1, r_2, \dots, r_n)$ and $U_1 = \operatorname{diag}(u_1, u_2, \dots, u_n)$. By construction, R_1 is Hermitian, U_1 is unitary, and $U_1 R_1 = B$.

Now consider the general case. If an $n \times n$ matrix B is normal then there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of B. It follows that $B = QDQ^{-1}$, where Q is a unitary matrix (transition matrix from the orthonormal basis of eigenvectors of B to the standard basis) and D is diagonal. By the above, $D = U_1 R_1$, where R_1 is an Hermitian matrix and U_1 is a unitary matrix. Let $R = QR_1Q^{-1}$ and $U = QU_1Q^{-1}$. Then $UR = QU_1Q^{-1}QR_1Q^{-1} = QU_1R_1Q^{-1} = QDQ^{-1} = B$. Since Q is unitary, we have $R^* = (QR_1Q^{-1})^* = (QR_1Q^*)^* = (Q^*)^*R_1^*Q^* = QR_1Q^* = QR_1Q^{-1} = R$ so that the matrix R is Hermitian. Similarly, $U^* = QU_1^*Q^{-1} = QU_1^{-1}Q^{-1} = (QU_1Q^{-1})^{-1} = U^{-1}$ so that the matrix U is unitary.

(ii) Find a symmetric matrix R_0 (with real entries) and an orthogonal matrix U_0 of the same dimensions such that U_0R_0 is not a normal matrix.

Solution: $R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $U_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Suppose $B_0 = U_0R_0$, where R_0 is a symmetric matrix and U_0 is an orthogonal matrix. Then

$$B_0^*B_0 = (U_0R_0)^*U_0R_0 = R_0^*U_0^*U_0R_0 = R_0U_0^{-1}U_0R_0 = R_0^2,$$
$$B_0B_0^* = U_0R_0(U_0R_0)^* = U_0R_0R_0^*U_0^* = U_0R_0^2U_0^{-1}.$$

Hence B_0 is normal if and only if U_0 commutes with R_0^2.