MATH 423

Linear Algebra II

Lecture 1:
Classical vectors.
Vector space.

Classical vectors

Vector is a mathematical concept characterized by its *magnitude* and *direction*.

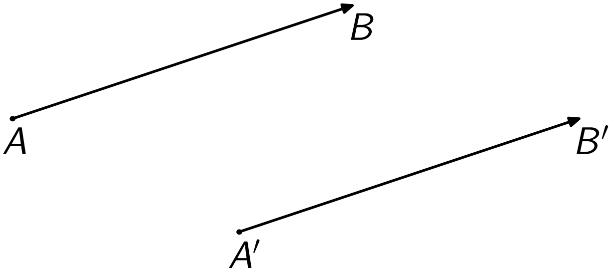
Scalar is a mathematical concept characterized by its *magnitude* and, possibly, *sign*.

Scalar is a real number (positive or negative).

Many physical quantities are vectors:

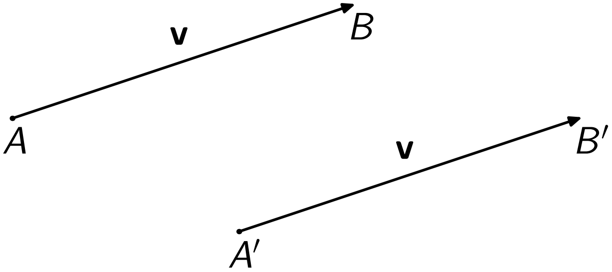
- force;
- displacement, velocity, acceleration;
- electric field, magnetic field.

Classical vectors: geometric approach



- A vector is represented by a directed segment.
- Directed segment is drawn as an arrow.
- Different arrows represent the same vector if they are of the same length and direction.

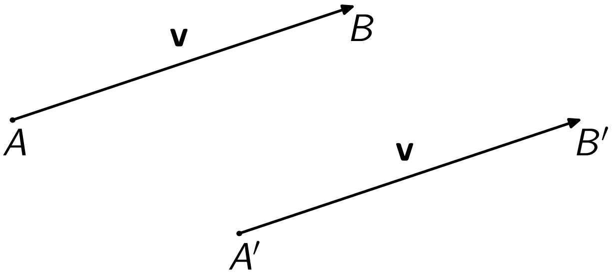
Classical vectors: geometric approach



$\overrightarrow{AB}$ denotes a vector represented by the arrow with tip (endpoint) at B and tail (beginning) at A.

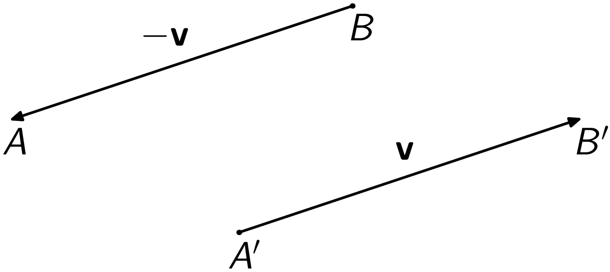
For any vector $\mathbf{v}$ and a point A there is a unique point B such that $\overrightarrow{AB} = \mathbf{v}$.

Classical vectors: geometric approach



Geometric fact: if a quadrilateral $ABB'A'$ is a parallelogram then $\overrightarrow{AB} = \overrightarrow{A'B'}$. The converse holds if the points A, B, A', B' are not on the same line.

Classical vectors: geometric approach

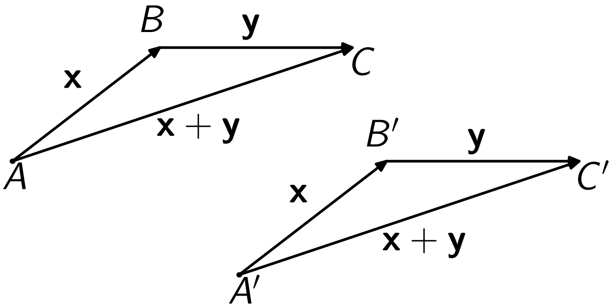


If $\mathbf{v} = \overrightarrow{AB}$ then $\overrightarrow{BA}$ is called the *negative vector* of $\mathbf{v}$ and denoted $-\mathbf{v}$.
$\overrightarrow{AA}$ is called the *zero vector* and denoted $\mathbf{0}$.

Vector addition

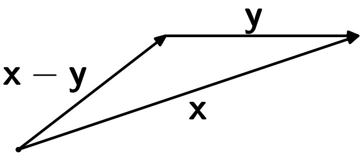
Given vectors $\mathbf{x}$ and $\mathbf{y}$, their *sum* $\mathbf{x}+\mathbf{y}$ is defined by the rule $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$.

That is, choose points A, B, C so that $\overrightarrow{AB}=\mathbf{x}$ and $\overrightarrow{BC}=\mathbf{y}$. Then $\mathbf{x}+\mathbf{y}=\overrightarrow{AC}$.



Vector subtraction

The *difference* of vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as $\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y})$.



Properties of vector addition:

$(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ (associative law)

$\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ (commutative law)

$\mathbf{x}+\mathbf{0}=\mathbf{0}+\mathbf{x}=\mathbf{x}$

$\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$

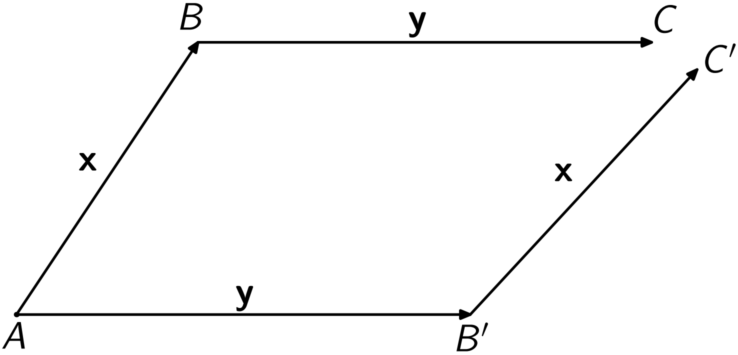
Let $\overrightarrow{AB}=\mathbf{x}$. Then $\mathbf{x}+\mathbf{0}=\overrightarrow{AB}+\overrightarrow{BB}=\overrightarrow{AB}=\mathbf{x}$,
$\mathbf{x}+(-\mathbf{x})=\overrightarrow{AB}+\overrightarrow{BA}=\overrightarrow{AA}=\mathbf{0}$.

Let $\overrightarrow{AB}=\mathbf{x}$, $\overrightarrow{BC}=\mathbf{y}$, and $\overrightarrow{CD}=\mathbf{z}$. Then
$(\mathbf{x}+\mathbf{y})+\mathbf{z}=(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}=\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$,
$\mathbf{x}+(\mathbf{y}+\mathbf{z})=\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$.

Parallelogram law

Let $\overrightarrow{AB} = \mathbf{x}$, $\overrightarrow{BC} = \mathbf{y}$, $\overrightarrow{AB'} = \mathbf{y}$, and $\overrightarrow{B'C'} = \mathbf{x}$.

Then $\mathbf{x} + \mathbf{y} = \overrightarrow{AC}$, $\mathbf{y} + \mathbf{x} = \overrightarrow{AC'}$.



Wrong picture!

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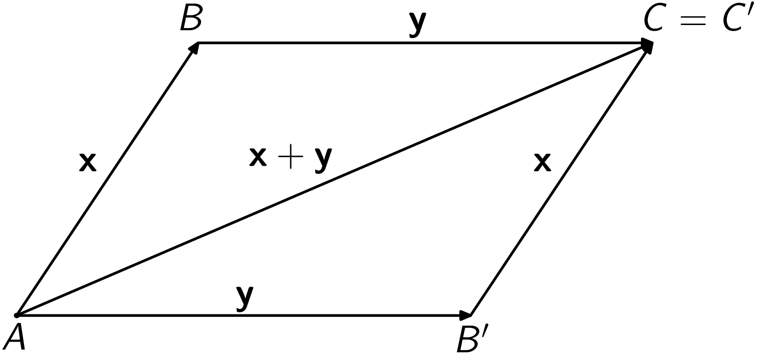
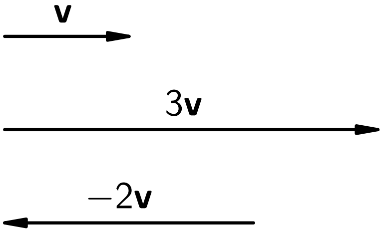


Right picture!

Scalar multiplication

Let $\mathbf{v}$ be a vector and $r \in \mathbb{R}$. By definition, $r\mathbf{v}$ is a vector whose magnitude is $|r|$ times the magnitude of $\mathbf{v}$. The direction of $r\mathbf{v}$ coincides with that of $\mathbf{v}$ if $r > 0$. If $r < 0$ then the directions of $r\mathbf{v}$ and $\mathbf{v}$ are opposite.



Scalar multiplication

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Properties of scalar multiplication:

$r(s\mathbf{x}) = (rs)\mathbf{x}$ (associative law)

$r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$ (distributive law #1)

$(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$ (distributive law #2)

$1\mathbf{x} = \mathbf{x}$

$0\mathbf{x} = \mathbf{0}$

Classical vectors: algebraic approach

An n-dimensional coordinate vector is an element of $\mathbb{R}^n$, i.e., an ordered list $(x_1, x_2, \ldots, x_n)$ of n real numbers. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

Scalar multiple: $r\mathbf{x} = (rx_1, rx_2, \ldots, rx_n)$

Zero vector: $\mathbf{0} = (0, 0, \ldots, 0)$

Additive inverse: $-\mathbf{y} = (-y_1, -y_2, \ldots, -y_n)$

Vector difference:

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$$

Properties of vector addition and scalar multiplication:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

$$\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$

$$\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$$

$$(rs)\mathbf{x} = r(s\mathbf{x})$$

$$r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$$

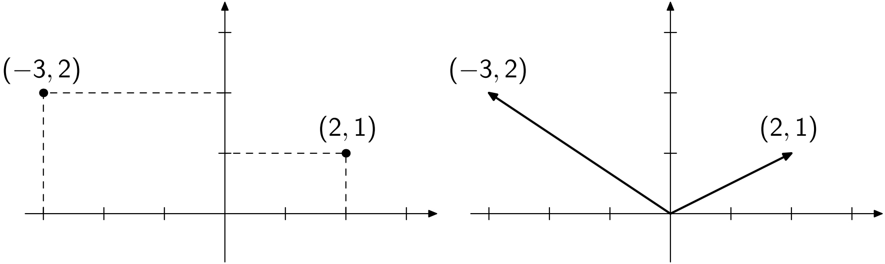
$$(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$$

$$1\mathbf{x} = \mathbf{x}$$

$$0\mathbf{x} = \mathbf{0}$$

$$(-1)\mathbf{x} = -\mathbf{x}$$

Cartesian coordinates: geometric meets algebraic



Cartesian coordinates allow us to identify a plane with $\mathbb{R}^2$ (similarly, a line with $\mathbb{R}$ and space with $\mathbb{R}^3$). Once we specify an *origin* O, each point A is associated a *position vector* $\overrightarrow{OA}$. Conversely, every vector has a unique representative with tail at O.

Abstract vector space: informal description

Vector space = *linear space* = a set V of objects (called *vectors*) that can be added and scaled.

That is, for any $\mathbf{u}, \mathbf{v} \in V$ and $r \in \mathbb{R}$ expressions

$$\boxed{\mathbf{u} + \mathbf{v}} \text{ and } \boxed{r\mathbf{u}}$$

should make sense.

Certain restrictions apply. For instance,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},$$
$$2\mathbf{u} + 3\mathbf{u} = 5\mathbf{u}.$$

That is, we want the addition and scalar multiplication in V to be like those of the classical vectors.

Abstract vector space: definition

Vector space is a set V equipped with two operations $\alpha : V \times V \to V$ and $\mu : \mathbb{R} \times V \to V$ that have certain properties (listed below).

The operation α is called *addition*. For any $\mathbf{u}, \mathbf{v} \in V$, the element $\alpha(\mathbf{u}, \mathbf{v})$ is denoted $\mathbf{u} + \mathbf{v}$.

The operation μ is called *scalar multiplication*. For any $r \in \mathbb{R}$ and $\mathbf{u} \in V$, the element $\mu(r, \mathbf{u})$ is denoted $r\mathbf{u}$.

Properties of addition and scalar multiplication (brief)

VS1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

VS2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$

VS3. $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$

VS4. $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$

VS5. $1\mathbf{x} = \mathbf{x}$

VS6. $(rs)\mathbf{x} = r(s\mathbf{x})$

VS7. $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$

VS8. $(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$

Properties of addition and scalar multiplication (detailed)

VS1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for all $\mathbf{x},\mathbf{y}\in V$.

VS2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for all $\mathbf{x},\mathbf{y},\mathbf{z}\in V$.

VS3. There exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{x}+\mathbf{0}=\mathbf{0}+\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}\in V$.

VS4. For any $\mathbf{x}\in V$ there exists an element of V, denoted $-\mathbf{x}$, such that $\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$.

VS5. $1\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}\in V$.

VS6. $(rs)\mathbf{x}=r(s\mathbf{x})$ for all $r,s\in\mathbb{R}$ and $\mathbf{x}\in V$.

VS7. $r(\mathbf{x}+\mathbf{y})=r\mathbf{x}+r\mathbf{y}$ for all $r\in\mathbb{R}$ and $\mathbf{x},\mathbf{y}\in V$.

VS8. $(r+s)\mathbf{x}=r\mathbf{x}+s\mathbf{x}$ for all $r,s\in\mathbb{R}$ and $\mathbf{x}\in V$.