MATH 423

Linear Algebra II

Lecture 4: Span. Spanning set. Linear independence.

Vector space over a field

The definition of a vector space V over an arbitrary field $\mathbb{F}$ is obtained from the definition of the usual vector space by changing $\mathbb{R}$ to $\mathbb{F}$ everywhere in the latter. Namely, the changes are:

- scalar multiple $r\mathbf{x}$ is defined for all $r \in \mathbb{F}$ and $\mathbf{x} \in V$.
- VS6. $(rs)\mathbf{x} = r(s\mathbf{x})$ for all $r, s \in \mathbb{F}$ and $\mathbf{x} \in V$.
- VS7. $r(\mathbf{x}+\mathbf{y}) = r\mathbf{x}+r\mathbf{y}$ for all $r \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y} \in V$.
- VS8. $(r+s)\mathbf{x} = r\mathbf{x}+s\mathbf{x}$ for all $r, s \in \mathbb{F}$ and $\mathbf{x} \in V$.

In what follows, it is okay to assume that $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Theorem A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{F}.$$

Remarks. The zero vector in a subspace is the same as the zero vector in V. Also, the subtraction in a subspace agrees with that in V.

Let V be a vector space (over a field $\mathbb{F}$). For any $\mathbf{v} \in V$ we denote by $\mathbb{F}\mathbf{v}$ the set of all scalar multiples of the vector $\mathbf{v}$ in V: $\mathbb{F}\mathbf{v} = \{r\mathbf{v} \mid r \in \mathbb{F}\}$.

Theorem 1 $\mathbb{F}\mathbf{v}$ is a subspace of V.

Proof: The set $\mathbb{F}\mathbf{v}$ is not empty since $\mathbb{F}$ is not empty.
$\mathbb{F}\mathbf{v}$ is closed under addition since $r\mathbf{v} + s\mathbf{v} = (r+s)\mathbf{v}$.
$\mathbb{F}\mathbf{v}$ is closed under scaling since $s(r\mathbf{v}) = (sr)\mathbf{v}$.

Given two subsets X and Y of V, we define another subset, denoted $X+Y$, by $X+Y=\{\mathbf{x}+\mathbf{y} \mid \mathbf{x}\in X,\ \mathbf{y}\in Y\}$.

Theorem 2 If X and Y are subspaces of V, then $X+Y$ is also a subspace of V.

Proof: The set $X+Y$ is not empty since X and Y are not empty. $X+Y$ is closed under addition since X and Y are:

$$(\mathbf{x}+\mathbf{y})+(\mathbf{x}'+\mathbf{y}')=(\mathbf{x}+\mathbf{x}')+(\mathbf{y}+\mathbf{y}').$$

$X+Y$ is closed under scaling since X and Y are:

$$r(\mathbf{x}+\mathbf{y})=r\mathbf{x}+r\mathbf{y}.$$

For any subsets $X_1, X_2, \ldots, X_n$ of V we define another subset

$$X_1 + X_2 + \cdots + X_n = \{\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n \mid \mathbf{x}_i \in X_i,\ 1 \leq i \leq n\}.$$

Theorem 3 The set $X_1 + X_2 + \cdots + X_n$ is a subspace of V provided that each X_i is a subspace of V.

Theorem 3 is proved by repeatedly applying Theorem 2. First $X_1 + X_2$ is a subspace. Then $X_1 + X_2 + X_3 = (X_1 + X_2) + X_3$ is a subspace. Then $X_1 + X_2 + X_3 + X_4 = (X_1 + X_2 + X_3) + X_4$ is a subspace, and so on.

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$.
Consider the set L of all linear combinations
$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{F}$.

Theorem 4 L is a subspace of V.

Proof: First of all, L is not empty. For example, $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$ belongs to L.

The set L is closed under addition since

$$(r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_n\mathbf{v}_n) + (s_1\mathbf{v}_1+s_2\mathbf{v}_2+\cdots+s_n\mathbf{v}_n) =$$
$$= (r_1+s_1)\mathbf{v}_1 + (r_2+s_2)\mathbf{v}_2 + \cdots + (r_n+s_n)\mathbf{v}_n.$$

The set L is closed under scalar multiplication since

$$t(r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_n\mathbf{v}_n) = (tr_1)\mathbf{v}_1+(tr_2)\mathbf{v}_2+\cdots+(tr_n)\mathbf{v}_n.$$

Alternative proof: It is easy to see that

$$L = \mathbb{F}\mathbf{v}_1 + \mathbb{F}\mathbf{v}_2 + \cdots + \mathbb{F}\mathbf{v}_n.$$

The previous theorems imply that L is a subspace.

Span: implicit definition

Let S be a subset of a vector space V.

Definition. The **span** of the set S, denoted $\mathrm{Span}(S)$, is the smallest subspace of V that contains S. That is,

- $\mathrm{Span}(S)$ is a subspace of V;
- for any subspace $W \subset V$ one has
$$S \subset W \implies \mathrm{Span}(S) \subset W.$$

Remark. The span of any set $S \subset V$ is well defined (it is the intersection of all subspaces of V that contain S).

Span: effective description

Let S be a subset of a vector space V.

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{F}$.

- If S is an infinite set then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k$, where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in S$ and $r_1, r_2, \dots, r_k \in \mathbb{F}$ $(k \geq 1)$.

- If S is the empty set then $\mathrm{Span}(S) = \{\mathbf{0}\}$.

Spanning set

Definition. A subset S of a vector space V is called a **spanning set** for V if $\mathrm{Span}(S) = V$.

Examples.

- Vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ form a spanning set for $\mathbb{F}^3$ as

$$(x,y,z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a spanning set for $\mathcal{M}_{2,2}(\mathbb{F})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{F}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Examples of linear independence

- Vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ in $\mathbb{R}^3$.

$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{0} \implies (x,y,z) = \mathbf{0}$
$\implies x = y = z = 0$

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O$
$\implies a = b = c = d = 0$

Theorem The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a linear combination of the other $k-1$ vectors.

Proof: **(i)** $\Longrightarrow$ **(ii)** Suppose that

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some $1 \leq i \leq k$. Then

$$\mathbf{v}_i = -\tfrac{r_1}{r_i}\mathbf{v}_1 - \cdots - \tfrac{r_{i-1}}{r_i}\mathbf{v}_{i-1} - \tfrac{r_{i+1}}{r_i}\mathbf{v}_{i+1} - \cdots - \tfrac{r_k}{r_i}\mathbf{v}_k.$$

(ii) $\Longrightarrow$ **(i)** Suppose that

$$\mathbf{v}_i = s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k$$

for some scalars s_j. Then

$$s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k = \mathbf{0}.$$