MATH 423

Linear Algebra II

Lecture 5:
Linear independence (continued).

Span

Let V be a vector space over a field $\mathbb{F}$ and let S be a subset of V.

Definition. The **span** of the set S, denoted $\mathrm{Span}(S)$, is the smallest subspace $W \subset V$ that contains S.

Theorem If S is not empty then $\mathrm{Span}(S)$ consists of all linear combinations $\boxed{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k}$ such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in S$ and $r_1, r_2, \ldots, r_k \in \mathbb{F}$.

In the case $\mathrm{Span}(S) = V$, we say that the set S **spans** the space V, or that S **generates** V, or that S is a **spanning set** for V.

Properties of span

Let S_0 and S be subsets of a vector space V.

- $S_0 \subset S \implies \mathrm{Span}(S_0) \subset \mathrm{Span}(S)$.
- $\mathrm{Span}(S_0) = V$ and $S_0 \subset S \implies \mathrm{Span}(S) = V$.
- If $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ is a spanning set for V and $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is also a spanning set for V.

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then
$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = (t_0 r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0 r_k + t_k)\mathbf{v}_k$.

- $\mathrm{Span}(S_0 \cup \{\mathbf{v}_0\}) = \mathrm{Span}(S_0)$ if and only if $\mathbf{v}_0 \in \mathrm{Span}(S_0)$.

If $\mathbf{v}_0 \in \mathrm{Span}(S_0)$, then $S_0 \cup \mathbf{v}_0 \subset \mathrm{Span}(S_0)$, which implies $\mathrm{Span}(S_0 \cup \{\mathbf{v}_0\}) \subset \mathrm{Span}(S_0)$. On the other hand, $\mathrm{Span}(S_0) \subset \mathrm{Span}(S_0 \cup \{\mathbf{v}_0\})$.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \ldots, r_k \in \mathbb{F}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Examples of linear independence

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ in $\mathbb{F}^3$.

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathcal{M}_{2,2}(\mathbb{F})$.

- Polynomials $1, x, x^2, \dots, x^n, \dots$ in $\mathcal{P}$ (or in $\mathbb{F}[x]$).

- The numbers 1 and i are linearly independent in $\mathbb{C}$ regarded as a vector space over $\mathbb{R}$ (however they are linearly dependent if $\mathbb{C}$ is regarded as a complex vector space).

- The empty set is always linearly independent.

Properties of linear independence

Let S_0 and S be subsets of a vector space V.

- If $S_0 \subset S$ and S is linearly independent, then so is S_0.
- If $S_0 \subset S$ and S_0 is linearly dependent, then so is S.
- If S is linearly independent in V and V is a subspace of W, then S is linearly independent in W.
- Any set containing $\mathbf{0}$ is linearly dependent.
- Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other $k-1$ vectors.
- Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if one of them is a scalar multiple the other.
- Two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if either of them is a scalar multiple the other.
- If S_0 is linearly independent and $\mathbf{v}_0 \in V \setminus S_0$ then $S_0 \cup \{\mathbf{v}_0\}$ is linearly independent if and only if $\mathbf{v}_0 \notin \mathrm{Span}(S)$.

Problem. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Solution: Suppose that $af_1(x) + bf_2(x) + cf_3(x) = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Let us differentiate this identity:

$$ax + bxe^x + ce^{-x} = 0,$$
$$a + be^x + bxe^x - ce^{-x} = 0,$$
$$2be^x + bxe^x + ce^{-x} = 0,$$
$$3be^x + bxe^x - ce^{-x} = 0,$$
$$4be^x + bxe^x + ce^{-x} = 0.$$

(the 5th identity)$-$(the 3rd identity): $2be^x = 0 \implies b = 0$.

Substitute $b = 0$ in the 3rd identity: $ce^{-x} = 0 \implies c = 0$.

Substitute $b = c = 0$ in the 2nd identity: $a = 0$.

Problem. Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Alternative solution: Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by xe^x:

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

The left-hand side approaches b as $x \to +\infty$. $\implies b = 0$

Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the identity by x:

$$a + cx^{-1}e^{-x} = 0.$$

The left-hand side approaches a as $x \to +\infty$. $\implies a = 0$

Now $ce^{-x} = 0 \implies c = 0$.

Linear independence over $\mathbb{Q}$

Since the set $\mathbb{R}$ of real numbers and the set $\mathbb{Q}$ of rational numbers are fields, we can regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Real numbers $r_1, r_2, \dots, r_n$ are said to be **linearly independent over** $\mathbb{Q}$ if they are linearly independent as vectors in that vector space.

Example. 1 and $\sqrt{2}$ are linearly independent over $\mathbb{Q}$.

Assume $a \cdot 1 + b\sqrt{2} = 0$ for some $a, b \in \mathbb{Q}$. We have to show that $a = b = 0$.
Indeed, $b = 0$ as otherwise $\sqrt{2} = -a/b$, a rational number. Then $a = 0$ as well.

In general, two nonzero real numbers r_1 and r_2 are linearly independent over $\mathbb{Q}$ if r_1/r_2 is irrational.

Linear independence over $\mathbb{Q}$

Example. 1, $\sqrt{2}$, and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$.

Assume $a + b\sqrt{2} + c\sqrt{3} = 0$ for some $a, b, c \in \mathbb{Q}$.
We have to show that $a = b = c = 0$.

$$a + b\sqrt{2} + c\sqrt{3} = 0 \implies a + b\sqrt{2} = -c\sqrt{3}$$
$$\implies (a + b\sqrt{2})^2 = (-c\sqrt{3})^2$$
$$\implies (a^2 + 2b^2 - 3c^2) + 2ab\sqrt{2} = 0.$$

Since 1 and $\sqrt{2}$ are linearly independent over $\mathbb{Q}$, we obtain $a^2 + 2b^2 - 3c^2 = 2ab = 0$. In particular, $a = 0$ or $b = 0$.

Then $a + c\sqrt{3} = 0$ or $b\sqrt{2} + c\sqrt{3} = 0$. However 1 and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$ as well as $\sqrt{2}$ and $\sqrt{3}$. Thus $a = b = c = 0$.