MATH 423

Linear Algebra II

# Lecture 6: Basis and dimension.

# Basis

*Definition.* Let $V$ be a vector space. A linearly independent spanning set for $V$ is called a **basis**.

**Theorem** A nonempty set $S \subset V$ is a basis for $V$ if and only if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are distinct vectors from $S$ and $r_1, \ldots, r_k \in \mathbb{F}$.

*Remark on uniqueness.* Expansions $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{v} = -\mathbf{v}_2 + 2\mathbf{v}_1$, and $\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2 + 0\mathbf{v}_3$ are considered the same.

**Theorem** A nonempty set $S \subset V$ is a basis for $V$ if and only if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from $S$ and $r_1, \dots, r_k \in \mathbb{F}$.

*Proof (***"if"***):* Assume that any vector in $V$ admits a unique expansion as described above. Then $\mathrm{Span}(S) = V$ so that $S$ is a spanning set.

Further, suppose $\mathbf{0} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$ for some distinct vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$. Since we also have $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k$, the uniqueness implies $r_i = 0$, $1 \le i \le k$. Therefore $S$ is linearly independent.

Thus $S$ is a basis.

**Theorem** A nonempty set $S \subset V$ is a basis for $V$ if and only if any vector $\mathbf{v} \in V$ is *uniquely represented* as a linear combination $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from $S$ and $r_1, \dots, r_k \in \mathbb{F}$.

*Proof (***"only if"***):* Assume that $S$ is a basis. Since $S$ is a spanning set for $V$, any vector $\mathbf{v} \in V$ admits an expansion $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from $S$ and $r_i \in \mathbb{F}$. Suppose that we also have $\mathbf{v} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_m\mathbf{u}_m$, for some distinct vectors $\mathbf{u}_1, \dots, \mathbf{u}_m \in S$ and some scalars $s_j \in \mathbb{F}$. Without loss of generality we can assume that $m = k$ and $\mathbf{u}_i = \mathbf{v}_i$, $1 \leq i \leq k$ (this can be achieved by adding terms of the form $0\mathbf{w}$ to both expansions and rearranging terms in one of them). Then

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k,$$

which implies $(r_1 - s_1)\mathbf{v}_1 + (r_2 - s_2)\mathbf{v}_2 + \cdots + (r_k - s_k)\mathbf{v}_k = \mathbf{0}$. Since the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent, we obtain $r_1 - s_1 = r_2 - s_2 = \ldots = r_k - s_k = 0$, i.e., the two expansions are the same.

*Examples.* • Standard basis for $\mathbb{F}^n$:
$\mathbf{e}_1 = (1, 0, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0, 0)$, $\ldots$,
$\mathbf{e}_n = (0, 0, 0, \ldots, 0, 1)$.

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
form a basis for $\mathcal{M}_{2,2}(\mathbb{F})$.

• Polynomials $1, x, x^2, \ldots, x^n$ form a basis for
$\mathcal{P}_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

• The empty set is a basis for the zero vector space $\{\mathbf{0}\}$.

# Dimension

**Theorem 1** Any vector space has a basis.

**Theorem 2** If a vector space $V$ has a finite basis, then all bases for $V$ are finite and have the same number of elements.

*Definition.* The **dimension** of a vector space $V$, denoted $\dim V$, (or $\dim_{\mathbb{F}} V$) is the number of elements in any of its bases.

*Examples.* • $\dim \mathbb{F}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{F})$: the space of $2\times 2$ matrices

$\dim \mathcal{M}_{2,2}(\mathbb{F}) = 4$

• $\mathcal{M}_{m,n}(\mathbb{F})$: the space of $m\times n$ matrices

$\dim \mathcal{M}_{m,n}(\mathbb{F}) = mn$

• $\mathcal{P}_n$: polynomials of degree at most $n$

$\dim \mathcal{P}_n = n + 1$

• $\mathcal{P}$: the space of all polynomials

$\dim \mathcal{P} = \infty$

• $\mathbb{C}$: complex numbers

$\dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{R}} \mathbb{C} = 2$

• $\{\mathbf{0}\}$: the trivial vector space

$\dim \{\mathbf{0}\} = 0$

**Problem.** Find the dimension of the plane $x + 2z = 0$ in $\mathbb{R}^3$.

The general solution of the equation $x + 2z = 0$ is

$$\begin{cases} x = -2s \\ y = t \\ z = s \end{cases} \qquad (t, s \in \mathbb{R})$$

That is, $(x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)$.

Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis so that the dimension of the plane is 2.

# Replacement Theorem

**Theorem** Suppose $S$ is a spanning set for a vector space $V$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent vectors in $V$. Then one can replace some $k$ vectors in $S$ by vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ so that the new set still spans $V$.

**Corollary 1** A linearly independent set cannot have more vectors than a spanning set.

**Corollary 2** If a vector space has a finite basis consisting of $n$ vectors, then

- any spanning set has at least $n$ vectors;
- any linearly independent set has at most $n$ vectors;
- any basis has exactly $n$ vectors.

## How to find a basis?

**Theorem** Let $S$ be a subset of a vector space $V$. Then the following conditions are equivalent:

**(i)** $S$ is a linearly independent spanning set for $V$, i.e., a basis;

**(ii)** $S$ is a minimal spanning set for $V$;

**(iii)** $S$ is a maximal linearly independent subset of $V$.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of $V$ to this set, and it will become linearly dependent".

## Part of the proof: (ii) $\Longrightarrow$ (i), (iii) $\Longrightarrow$ (i)

**Lemma 1** If a set $S$ is linearly dependent then one of vectors in $S$ is a linear combination of the others, or else $S = \{\mathbf{0}\}$.

**Lemma 2** Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ be a spanning set for a vector space $V$. If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for $V$.

**(ii) $\Longrightarrow$ (i):** If a spanning set $S$ is not a basis, it is linearly dependent. By Lemma 1, some $\mathbf{v} \in S$ is a linear combination of the other vectors in $S$, or else $S = \{\mathbf{0}\}$. In the first case, $S \setminus \{\mathbf{v}\}$ is a spanning set by Lemma 2. In the second case, $V = \{\mathbf{0}\}$ so that the empty set is a spanning set. In either case, $S$ is not a minimal spanning set.

**(iii) $\Longrightarrow$ (i):** If a linearly independent set $S$ is not a basis, it is not a spanning set for $V$. Take any vector $\mathbf{v} \in V$ not in $\mathrm{Span}(S)$. Then the set $S \cup \{\mathbf{v}\}$ is linearly independent so that $S$ is not maximal.

## How to find a basis?

**Theorem** Let $V$ be a vector space. Then
**(i)** any spanning set for $V$ can be reduced to a minimal spanning set;
**(ii)** any linearly independent subset of $V$ can be extended to a maximal linearly independent set.

**Corollary** Any spanning set contains a basis while any linearly independent set is contained in a basis.

# How to find a basis?

*Approach 1.* Get a spanning set for the vector space, then reduce this set to a basis dropping one vector at a time.

**Theorem** A vector space has a finite basis whenever it has a finite spanning set.

*Proof:* Suppose $S$ is a finite spanning set for a vector space $V$. If $S$ is not a minimal spanning set, then we can drop one vector from $S$ so that the new set $S_1$ also spans $V$. If $S_1$ is still not minimal, we can drop one more vector to obtain yet another spanning set $S_2$. And so on... Since $S$ is a finite set, this inductive procedure will eventually produce a minimal spanning set, i.e., a basis for $V$.

# How to find a basis?

*Approach 2.* Build a maximal linearly independent set adding one vector at a time.

If the vector space $V$ is trivial, it has the empty basis. If $V \neq \{\mathbf{0}\}$, pick any vector $\mathbf{v}_1 \neq \mathbf{0}$. If $\mathbf{v}_1$ spans $V$, it is a basis. Otherwise pick any vector $\mathbf{v}_2 \in V$ that is not in the span of $\mathbf{v}_1$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ span $V$, they constitute a basis. Otherwise pick any vector $\mathbf{v}_3 \in V$ that is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. And so on...

*Modifications.* Instead of the empty set, we can start with any linearly independent set (if we are given one). If we are given a spanning set $S$, it is enough to pick new vectors only in $S$.

*Remark.* This inductive procedure works for finite-dimensional vector spaces. There is an analogous procedure for infinite-dimensional spaces (**transfinite induction**).

Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent in $\mathbb{R}^3$.

**Problem.** Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

Since vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for $\mathbb{R}^3$, at least one of them can be chosen as $\mathbf{v}_3$.

One can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ are two bases for $\mathbb{R}^3$:

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1 \neq 0, \qquad \begin{vmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0.$$