MATH 423

Linear Algebra II

Lecture 7: Linear transformations. Range and null-space.

Linear mapping = linear transformation

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** (or $\mathbb{F}$-**linear**) if

$$\boxed{L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{F}$.

A linear mapping $\ell : V \to \mathbb{F}$ is called a **linear functional** on V.

If $V_1 = V_2$ (or if both V_1 and V_2 are functional spaces) then a linear mapping $L : V_1 \to V_2$ is called a **linear operator**.

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for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{F}$.

Remark. A function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ is a linear transformation of the vector space $\mathbb{R}$ only if $b = 0$.

Basic properties of linear mappings

Let $L : V_1 \to V_2$ be a linear mapping.

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1 L(\mathbf{v}_1) + \cdots + r_k L(\mathbf{v}_k)$

for all $k \geq 1$, $\mathbf{v}_1, \dots, \mathbf{v}_k \in V_1$, and $r_1, \dots, r_k \in \mathbb{F}$.

$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) = L(r_1\mathbf{v}_1) + L(r_2\mathbf{v}_2) = r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2)$,
$L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3) = L(r_1\mathbf{v}_1 + r_2\mathbf{v}_2) + L(r_3\mathbf{v}_3) =$
$= r_1 L(\mathbf{v}_1) + r_2 L(\mathbf{v}_2) + r_3 L(\mathbf{v}_3)$, and so on.

- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in

V_1 and V_2, respectively.

$L(\mathbf{0}_1) = L(0\mathbf{0}_1) = 0L(\mathbf{0}_1) = \mathbf{0}_2$.

- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

$L(-\mathbf{v}) = L((-1)\mathbf{v}) = (-1)L(\mathbf{v}) = -L(\mathbf{v})$.

Examples of linear mappings

- *Scaling* $L : V \to V$, $L(\mathbf{v}) = s\mathbf{v}$, where $s \in \mathbb{F}$.

$L(\mathbf{x}+\mathbf{y}) = s(\mathbf{x}+\mathbf{y}) = s\mathbf{x}+s\mathbf{y} = L(\mathbf{x})+L(\mathbf{y})$,
$L(r\mathbf{x}) = s(r\mathbf{x}) = (sr)\mathbf{x} = r(s\mathbf{x}) = rL(\mathbf{x})$.

- *Dot product with a fixed vector*

$\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(\mathbf{v}) = \mathbf{v}\cdot\mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^n$.

$\ell(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y})\cdot\mathbf{v}_0 = \mathbf{x}\cdot\mathbf{v}_0 + \mathbf{y}\cdot\mathbf{v}_0 = \ell(\mathbf{x})+\ell(\mathbf{y})$,
$\ell(r\mathbf{x}) = (r\mathbf{x})\cdot\mathbf{v}_0 = r(\mathbf{x}\cdot\mathbf{v}_0) = r\ell(\mathbf{x})$.

- *Cross product with a fixed vector*

$L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{v}) = \mathbf{v}\times\mathbf{v}_0$, where $\mathbf{v}_0 \in \mathbb{R}^3$.

- *Multiplication by a fixed matrix*

$L : \mathbb{R}^n \to \mathbb{R}^m$, $L(\mathbf{v}) = A\mathbf{v}$, where A is an $m\times n$ matrix and all vectors are column vectors.

Linear mappings of functional vector spaces

- *Evaluation at a fixed point*

$\ell : \mathcal{F}(S) \to \mathbb{R},\ \ \ell(f) = f(a),$ where $a \in S$.

- *Multiplication by a fixed function*

$L : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}),\ \ L(f) = gf,$ where $g \in \mathcal{F}(\mathbb{R})$.

- *Differentiation* $D : C^1(\mathbb{R}) \to C(\mathbb{R}),\ \ L(f) = f'$.

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g),$
$D(rf) = (rf)' = rf' = rD(f).$

- *Integration over a finite interval*

$\ell : C(\mathbb{R}) \to \mathbb{R},\ \ \ell(f) = \int_a^b f(x)\,dx,$ where $a, b \in \mathbb{R},\ \ a < b$.

$\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices.

- $\alpha : \mathcal{M}_{m,n}(\mathbb{R}) \to \mathcal{M}_{n,m}(\mathbb{R})$, $\alpha(A) = A^t$, transpose of A.

$\alpha(A+B) = \alpha(A) + \alpha(B) \iff (A+B)^t = A^t + B^t$.
$\alpha(rA) = r\,\alpha(A) \iff (rA)^t = rA^t$.
Hence α is linear.

- $\beta : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}$, $\beta(A) = \det A$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We have $\det(A) = \det(B) = 0$ while $\det(A+B) = 1$.
Hence $\beta(A+B) \neq \beta(A) + \beta(B)$ so that β is not linear.

More properties of linear mappings

- If a linear mapping $L : V \to W$ is invertible then the inverse mapping $L^{-1} : W \to V$ is also linear.

Given vectors $\mathbf{w}_1, \mathbf{w}_2 \in W$, let $\mathbf{v}_1 = L^{-1}(\mathbf{w}_1)$, $\mathbf{v}_2 = L^{-1}(\mathbf{w}_2)$.
Since L is linear, $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2$.
That is, $L^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v}_1 + \mathbf{v}_2 = L^{-1}(\mathbf{w}_1) + L^{-1}(\mathbf{w}_2)$.
Given a vector $\mathbf{w} \in W$, let $\mathbf{v} = L^{-1}(\mathbf{w})$. Since L is linear, for any scalar r we have $L(r\mathbf{v}) = rL(\mathbf{v}) = r\mathbf{w}$. That is,
$L^{-1}(r\mathbf{w}) = r\mathbf{v} = rL^{-1}(\mathbf{w})$.

- If $L : V \to W$ and $M : W \to X$ are linear mappings then the composition $M \circ L : V \to X$ is also linear.

$$(M\circ L)(\mathbf{v}_1 + \mathbf{v}_2) = M\big(L(\mathbf{v}_1 + \mathbf{v}_2)\big) = M\big(L(\mathbf{v}_1) + L(\mathbf{v}_2)\big)$$
$$= M(L(\mathbf{v}_1)) + M(L(\mathbf{v}_2)) = (M\circ L)(\mathbf{v}_1) + (M\circ L)(\mathbf{v}_2).$$
$$(M\circ L)(r\mathbf{v}) = M(L(r\mathbf{v})) = M\big(r\,L(\mathbf{v})\big) = r\,M(L(\mathbf{v})).$$

Vector space of linear transformations

Let W be a vector space over a field $\mathbb{F}$. For any nonempty set S let $\mathcal{F}(S, W)$ denote the set of all mappings $f : S \to W$. The set $\mathcal{F}(S, W)$ is naturally endowed with the structure of a vector space over $\mathbb{F}$ (this was already done before in the case $W = \mathbb{R}$). Namely, for any functions $f, g \in \mathcal{F}(S, W)$ we define the sum $f + g$ by $(f + g)(x) = f(x) + g(x)$, $x \in S$. For any function $f \in \mathcal{F}(S, W)$ and scalar $r \in \mathbb{F}$ we define the scalar multiple rf by $(rf)(x) = r \cdot f(x)$, $x \in S$.

For any vector space V over $\mathbb{F}$ we denote by $\mathcal{L}(V, W)$ a subset of $\mathcal{F}(V, W)$ consisting of all linear transformations from V to W.

Theorem $\mathcal{L}(V, W)$ is a subspace of $\mathcal{F}(V, W)$.

Examples of linear differential operators

- an ordinary differential operator

$$L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2,$$

where g_0, g_1, g_2 are smooth functions on $\mathbb{R}$.
That is, $L(f) = g_0 f'' + g_1 f' + g_2 f$.

- Laplace's operator $\Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

(a.k.a. the Laplacian; also denoted by ∇^2).

Range and null-space

Let V, W be vector spaces and $L : V \to W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $\mathcal{R}(L)$ (or $L(V)$).

The **null-space** (or **kernel**) of L, denoted $\mathcal{N}(L)$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem **(i)** The range $\mathcal{R}(L)$ is a subspace of W.
(ii) The null-space $\mathcal{N}(L)$ is a subspace of V.

$\dim \mathcal{R}(L)$ is called the **rank** of the transformation L.
$\dim \mathcal{N}(L)$ is called the **nullity** of L.

Dimension Theorem

Theorem Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space V to a vector space W. Then $\boxed{\dim \mathcal{R}(L) + \dim \mathcal{N}(L) = \dim V.}$

The null-space $\mathcal{N}(L)$ is a subspace of V. It is finite-dimensional since the vector space V is.

Take a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ for the subspace $\mathcal{N}(L)$, then extend it to a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ for the entire space V.

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Assuming the claim is proved, we obtain

$$\dim \mathcal{R}(L) = m, \quad \dim \mathcal{N}(L) = k, \quad \dim V = k + m.$$

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Proof (spanning): Any vector $\mathbf{w} \in \mathcal{R}(L)$ is represented as $\mathbf{w} = L(\mathbf{v})$, where $\mathbf{v} \in V$. Then

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_m\mathbf{u}_m$$

for some $\alpha_i, \beta_j \in \mathbb{F}$. It follows that

$$\mathbf{w} = L(\mathbf{v}) = \alpha_1 L(\mathbf{v}_1) + \cdots + \alpha_k L(\mathbf{v}_k) + \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m)$$

$$= \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m).$$

Note that $L(\mathbf{v}_i) = \mathbf{0}$ since $\mathbf{v}_i \in \mathcal{N}(L)$.
Thus $\mathcal{R}(L)$ is spanned by the vectors $L(\mathbf{u}_1), \dots, L(\mathbf{u}_m)$.

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_m)$ form a basis for the range of L.

Proof (linear independence): Assume that

$$t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0}$$

for some $t_i \in \mathbb{F}$. Let $\mathbf{u} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m$. Since

$$L(\mathbf{u}) = t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0},$$

the vector $\mathbf{u}$ belongs to the null-space of L. Therefore $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k$ for some $s_j \in \mathbb{F}$. It follows that

$$t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m - s_1\mathbf{v}_1 - s_2\mathbf{v}_2 - \cdots - s_k\mathbf{v}_k = \mathbf{u} - \mathbf{u} = \mathbf{0}.$$

Linear independence of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_1, \ldots, \mathbf{u}_m$ implies that $t_1 = \cdots = t_m = 0$ (as well as $s_1 = \cdots = s_k = 0$). Thus the vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_m)$ are linearly independent.