MATH 423

Linear Algebra II

Lecture 8:
Subspaces and linear transformations.
Basis and coordinates.
Matrix of a linear transformation.

Linear transformation

Definition. Given vector spaces V_1 and V_2 over a field $\mathbb{F}$, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$

$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{F}$.

Basic properties of linear mappings:

- $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1L(\mathbf{v}_1) + \cdots + r_kL(\mathbf{v}_k)$ for all $k \geq 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{F}$.
- $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.
- $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Subspaces and linear maps

Let V_1, V_2 be vector spaces and $L : V_1 \to V_2$ be a linear map. Given a set $U \subset V_1$, its **image** under the map L, denoted $L(U)$, is the set of all vectors in V_2 that can be represented as $L(\mathbf{x})$ for some $\mathbf{x} \in U$.

Theorem If U is a subspace of V_1 then $L(U)$ is a subspace of V_2.

Proof: U is nonempty $\implies$ $L(U)$ is nonempty.

Let $\mathbf{u}, \mathbf{v} \in L(U)$. This means $\mathbf{u} = L(\mathbf{x})$ and $\mathbf{v} = L(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in U$. By linearity, $\mathbf{u} + \mathbf{v} = L(\mathbf{x}) + L(\mathbf{y}) = L(\mathbf{x} + \mathbf{y})$. Since U is a subspace of V_1, we have $\mathbf{x} + \mathbf{y} \in U$ so that $\mathbf{u} + \mathbf{v} \in L(U)$.

Similarly, if $\mathbf{u} = L(\mathbf{x})$ for some $\mathbf{x} \in U$ then for any $r \in \mathbb{F}$ we have $r\mathbf{u} = rL(\mathbf{x}) = L(r\mathbf{x}) \in L(U)$.

Subspaces and linear maps

Let V_1, V_2 be vector spaces and $L : V_1 \to V_2$ be a linear map. Given a set $W \subset V_2$, its **preimage** (or **inverse image**) under the map L, denoted $L^{-1}(W)$, is the set of vectors $\mathbf{x} \in V_1$ such that $L(\mathbf{x}) \in W$.

Theorem If W is a subspace of V_2 then its preimage $L^{-1}(W)$ is a subspace of V_1.

Proof: Let $\mathbf{0}_1$ be the zero vector in V_1 and $\mathbf{0}_2$ be the zero vector in V_2. By linearity, $L(\mathbf{0}_1) = \mathbf{0}_2$. Since W is a subspace of V_2, it contains $\mathbf{0}_2$. Hence $\mathbf{0}_1 \in L^{-1}(W)$.

Let $\mathbf{x}, \mathbf{y} \in L^{-1}(W)$. This means that $L(\mathbf{x}), L(\mathbf{y}) \in W$. Then $L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$ is in W since W is closed under addition. Therefore $\mathbf{x} + \mathbf{y} \in L^{-1}(W)$.

Similarly, if $L(\mathbf{x}) \in W$ for some $\mathbf{x} \in V_1$ then for any $r \in \mathbb{F}$ we have $L(r\mathbf{x}) = rL(\mathbf{x}) \in W$ so that $r\mathbf{x} \in L^{-1}(W)$.

Range and null-space

Let V, W be vector spaces and $L : V \to W$ be a linear mapping.

Definition. The **range** (or **image**) of L is the set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. The range of L is denoted $\mathcal{R}(L)$.

The **null-space** (or **kernel**) of L, denoted $\mathcal{N}(L)$, is the set of all vectors $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}$.

Theorem **(i)** The range $\mathcal{R}(L)$ is a subspace of W.
(ii) The null-space $\mathcal{N}(L)$ is a subspace of V.

Proof: $\mathcal{R}(L) = L(V)$, $\mathcal{N}(L) = L^{-1}(\{\mathbf{0}\})$.

Dimension Theorem

Theorem Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space V to a vector space W. Then $\boxed{\dim \mathcal{R}(L) + \dim \mathcal{N}(L) = \dim V.}$

The null-space $\mathcal{N}(L)$ is a subspace of V. It is finite-dimensional since the vector space V is.

Take a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ for the subspace $\mathcal{N}(L)$, then extend it to a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ for the entire space V.

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Assuming the claim is proved, we obtain

$$\dim \mathcal{R}(L) = m, \quad \dim \mathcal{N}(L) = k, \quad \dim V = k + m.$$

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Proof (spanning): Any vector $\mathbf{w} \in \mathcal{R}(L)$ is represented as $\mathbf{w} = L(\mathbf{v})$, where $\mathbf{v} \in V$. Then

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_m\mathbf{u}_m$$

for some $\alpha_i, \beta_j \in \mathbb{F}$. By linearity of L,

$$\mathbf{w} = L(\mathbf{v}) = \alpha_1 L(\mathbf{v}_1) + \cdots + \alpha_k L(\mathbf{v}_k) + \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m)$$

$$= \beta_1 L(\mathbf{u}_1) + \cdots + \beta_m L(\mathbf{u}_m).$$

Note that $L(\mathbf{v}_i) = \mathbf{0}$ since $\mathbf{v}_i \in \mathcal{N}(L)$.
Thus $\mathcal{R}(L)$ is spanned by the vectors $L(\mathbf{u}_1), \dots, L(\mathbf{u}_m)$.

Claim Vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ form a basis for the range of L.

Proof (linear independence): Assume that

$$t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0}$$

for some $t_i \in \mathbb{F}$. Let $\mathbf{u} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m$. Since

$$L(\mathbf{u}) = t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \cdots + t_m L(\mathbf{u}_m) = \mathbf{0},$$

the vector $\mathbf{u}$ belongs to the null-space of L. Therefore $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k$ for some $s_j \in \mathbb{F}$. It follows that

$$t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + \cdots + t_m\mathbf{u}_m - s_1\mathbf{v}_1 - s_2\mathbf{v}_2 - \cdots - s_k\mathbf{v}_k = \mathbf{u} - \mathbf{u} = \mathbf{0}.$$

Linear independence of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m$ implies that $t_1 = \cdots = t_m = 0$ (as well as $s_1 = \cdots = s_k = 0$). Thus the vectors $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ are linearly independent.

Let V_1, V_2 be vector spaces and $L : V_1 \to V_2$ be a linear map.

Definition. The map L is **one-to-one** if it maps different vectors from V_1 to different vectors in V_2. That is, for any $\mathbf{x}, \mathbf{y} \in V_1$ we have that $\mathbf{x} \neq \mathbf{y}$ implies $L(\mathbf{x}) \neq L(\mathbf{y})$.

The map L is **onto** if any element $\mathbf{y} \in V_2$ is represented as $L(\mathbf{x})$ for some $\mathbf{x} \in V_1$. If the map L is both one-to-one and onto, then the inverse map $L^{-1} : V_2 \to V_1$ is well defined.

Theorem A linear map L is one-to-one if and only if the nullspace $\mathcal{N}(L)$ is trivial.

Proof: Let $\mathbf{0}_1$ be the zero vector in V_1 and $\mathbf{0}_2$ be the zero vector in V_2. If a vector $\mathbf{x} \neq \mathbf{0}_1$ belongs to $\mathcal{N}(L)$, then $L(\mathbf{x}) = \mathbf{0}_2 = L(\mathbf{0}_1)$ so that L is not one-to-one.

Conversely, assume that $\mathcal{N}(L)$ is trivial. By linearity, $L(\mathbf{x} - \mathbf{y}) = L(\mathbf{x}) - L(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V_1$. Therefore $L(\mathbf{x}) = L(\mathbf{y}) \implies \mathbf{x} - \mathbf{y} \in \mathcal{N}(L) \implies \mathbf{x} = \mathbf{y}$. Thus L is one-to-one.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{F}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The **coordinate mapping**

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

establishes a one-to-one correspondence between V and $\mathbb{F}^n$. This correspondence is **linear**.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be elements of a vector space V. Define a map $f : \mathbb{F}^n \to V$ by

$$f(x_1, x_2, \dots, x_n) = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n.$$

Theorem **(i)** The map f linear.
(ii) If vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent then f is one-to-one.
(iii) If vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ span V then f is onto.
(iv) If vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ form a basis for V then f is one-to-one and onto.

Proof: The map f is linear since

$$(x_1+y_1)\mathbf{v}_1 + (x_2+y_2)\mathbf{v}_2 + \cdots + (x_n+y_n)\mathbf{v}_n$$
$$= (x_1\mathbf{v}_1+x_2\mathbf{v}_2+\cdots+x_n\mathbf{v}_n) + (y_1\mathbf{v}_1+y_2\mathbf{v}_2+\cdots+y_n\mathbf{v}_n),$$
$$(rx_1)\mathbf{v}_1+(rx_2)\mathbf{v}_2+\cdots+(rx_n)\mathbf{v}_n = r(x_1\mathbf{v}_1+x_2\mathbf{v}_2+\cdots+x_n\mathbf{v}_n)$$

for all $x_i, y_i, r \in \mathbb{F}$. Further, linear independence of $\mathbf{v}_1, \dots, \mathbf{v}_n$ means that the null-space of f is trivial, which is equivalent to f being one-to-one. Finally, statement (iii) is obvious while statement (iv) follows from (ii) and (iii).

Examples. • Coordinates of a vector $\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n$ relative to the standard basis $\mathbf{e}_1 = (1, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0, 0)$,..., $\mathbf{e}_n = (0, 0, \ldots, 0, 1)$ are $(x_1, x_2, \ldots, x_n)$.

• Coordinates of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{F})$ relative to the basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are (a, c, b, d).

• Coordinates of a polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{P}_n$ relative to the basis $1, x, x^2, \ldots, x^n$ are $(a_0, a_1, \ldots, a_n)$.

Matrix of a linear transformation

Let V, W be vector spaces and $L : V \to W$ be a linear map. Let $\alpha = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ be an ordered basis for V and $\beta = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]$ be an ordered basis for W.

Definition. The **matrix** of L relative to the bases α and β is an $m \times n$ matrix whose consecutive columns are coordinates of vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)$ relative to the basis β.

Notation. $[\mathbf{w}]_\beta$ denotes coordinates of $\mathbf{w}$ relative to the ordered basis β, regarded as a column vector. $[L]_\alpha^\beta$ denotes the matrix of L relative to α and β. Then

$$[L]_\alpha^\beta = \big([L(\mathbf{v}_1)]_\beta, [L(\mathbf{v}_2)]_\beta, \ldots, [L(\mathbf{v}_n)]_\beta\big).$$

Examples. • $D : \mathcal{P}_2 \to \mathcal{P}_1$, $(Dp)(x) = p'(x)$.
Let $\alpha = [1, x, x^2]$, $\beta = [1, x]$. Columns of the matrix $[D]_\alpha^\beta$ are coordinates of polynomials $D1$, Dx, Dx^2 w.r.t. the basis $1, x$.

$$D1 = 0,\ Dx = 1,\ Dx^2 = 2x \implies [D]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

• $L : \mathcal{P}_2 \to \mathcal{P}_2$, $(Lp)(x) = p(x+1)$.
Let us find the matrix $[L]_\alpha^\alpha$:
$L1 = 1$, $Lx = 1 + x$, $Lx^2 = (x+1)^2 = 1 + 2x + x^2$.

$$\implies [L]_\alpha^\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$