MATH 423

Linear Algebra II

Lecture 9:
Matrix of a linear transformation (continued).
Matrix multiplication.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{F}$. The coefficients $x_1, x_2, \dots, x_n$ are called the **coordinates** of **v** with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

The **coordinate mapping** $\mathbf{v} \mapsto (x_1, x_2, \dots, x_n)$ establishes a one-to-one correspondence between V and $\mathbb{F}^n$. This correspondence is linear.

Notation. $[\mathbf{v}]_\beta$ denotes coordinates of **v** relative to an ordered basis β, regarded as a column vector.

Matrix of a linear transformation

Let V, W be vector spaces and $L : V \to W$ be a linear map. Let $\alpha = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ be an ordered basis for V and $\beta = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]$ be an ordered basis for W.

Definition. The **matrix** of L relative to the bases α and β is an $m \times n$ matrix whose consecutive columns are coordinates of vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)$ relative to the basis β.

Notation. $[L]_\alpha^\beta$ denotes the matrix of L relative to the bases α and β. That is,

$$[L]_\alpha^\beta = \big([L(\mathbf{v}_1)]_\beta, [L(\mathbf{v}_2)]_\beta, \ldots, [L(\mathbf{v}_n)]_\beta\big).$$

If $V = W$ then $[L]_\alpha^\alpha$ is also denoted $[L]_\alpha$.

Let V and W be vector spaces and S be a subset of V.

Theorem 1 **(i)** If S spans V, then any linear transformation $L : V \to W$ is uniquely determined by its restriction to S.
(ii) If S is linearly independent then any function $L : S \to W$ can be extended to a linear transformation from V to W.
(iii) If S is a basis for V then any function $L : S \to W$ can be uniquely extended to a linear transformation from V to W.

Idea of the proof: If $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $\mathbf{v}_i \in S$, $r_i \in \mathbb{F}$, then $L(\mathbf{v}) = r_1L(\mathbf{v}_1) + r_2L(\mathbf{v}_2) + \cdots + r_nL(\mathbf{v}_n)$ for any linear map $L : V \to W$.

Theorem 2 Suppose $\alpha = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ is an ordered basis for V and $\beta = [\mathbf{w}_1, \dots, \mathbf{w}_m]$ is an ordered basis for W. Then a mapping $M : \mathcal{L}(V, W) \to \mathcal{M}_{m,n}(\mathbb{F})$ given by $M(L) = [L]_\alpha^\beta$ is linear and invertible (i.e., one-to-one and onto).

Scalar product

Definition. The **dot product** of n-dimensional vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$ is a scalar

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k .$$

The dot product is also called the **scalar product**.

Matrix multiplication

The product of matrices A and B with entries in a field $\mathbb{F}$ is defined if the number of columns in A matches the number of rows in B.

Definition. Let $A = (a_{ik})$ be an $m \times n$ matrix and $B = (b_{kj})$ be an $n \times p$ matrix. The **product** AB is defined to be the $m \times p$ matrix $C = (c_{ij})$ such that $\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$ for all indices i, j.

That is, matrices are multiplied **row by column**:

$$\begin{pmatrix} * & * & * \\ \boxed{*} & \boxed{*} & \boxed{*} \end{pmatrix} \begin{pmatrix} * & * & \boxed{*} & * \\ * & * & \boxed{*} & * \\ * & * & \boxed{*} & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & \boxed{*} & * \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \hline a_{21} & a_{22} & \dots & a_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

$$B = \left(\begin{array}{c|c|c|c} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{array}\right) = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p)$$

$$\implies AB = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{w}_p \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m \cdot \mathbf{w}_1 & \mathbf{v}_m \cdot \mathbf{w}_2 & \dots & \mathbf{v}_m \cdot \mathbf{w}_p \end{pmatrix}$$

Examples.

$$(x_1, x_2, \ldots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \left(\sum_{k=1}^{n} x_k y_k\right),$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} (x_1, x_2, \ldots, x_n) = \begin{pmatrix} y_1 x_1 & y_1 x_2 & \ldots & y_1 x_n \\ y_2 x_1 & y_2 x_2 & \ldots & y_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_n x_1 & y_n x_2 & \ldots & y_n x_n \end{pmatrix}.$$

Linear maps and matrix multiplication

Theorem 1 Suppose $\alpha = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is an ordered basis for V and $\beta = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ is an ordered basis for W. Then for any linear transformation $L : V \to W$ and any vector $\mathbf{v} \in V$,

$$[L(\mathbf{v})]_\beta = [L]_\alpha^\beta [\mathbf{v}]_\alpha.$$

Theorem 2 Suppose $\gamma = [\mathbf{x}_1, \ldots, \mathbf{x}_k]$ is an ordered basis for X. Then for any linear transformations $L : V \to W$ and $T : W \to X$,

$$[T \circ L]_\alpha^\gamma = [T]_\beta^\gamma [L]_\alpha^\beta.$$

Problem. Consider a linear operator L on the vector space of 2×2 matrices given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Find the matrix of L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let γ denote the ordered basis E_1, E_2, E_3, E_4.

It follows from the definition that $[L]_\gamma$ is a 4×4 matrix whose columns are coordinates of the matrices

$$L(E_1), L(E_2), L(E_3), L(E_4)$$

with respect to the basis E_1, E_2, E_3, E_4.

$$L(E_1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = 1E_1 + 0E_2 + 3E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = 0E_1 + 1E_2 + 0E_3 + 3E_4,$$

$$L(E_3) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = 2E_1 + 0E_2 + 4E_3 + 0E_4,$$

$$L(E_4) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = 0E_1 + 2E_2 + 0E_3 + 4E_4.$$

Therefore

$$[L]_\gamma = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.$$

Thus the relation

$$\begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

is equivalent to the relation

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Consider a linear operator $L : \mathcal{P}_2 \to \mathcal{P}_2$ given by $(Lp)(x) = p(x+1)$. In the previous lecture, it was found that the matrix of L relative to the basis $1, x, x^2$ was $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

This means that the polynomial identity

$$b_1 + b_2 x + b_3 x^2 = a_1 + a_2(x+1) + a_3(x+1)^2$$

is equivalent to the relation

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Matrix transformations

Any $m \times n$ matrix $A \in \mathcal{M}_{m,n}(\mathbb{F})$ gives rise to a transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$ given by $L_A(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{F}^n$ and $L(\mathbf{x}) \in \mathbb{F}^m$ are regarded as column vectors. This transformation is **linear**.

Example. $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

Let $\mathbf{e}_1 = (1,0,0)^t$, $\mathbf{e}_2 = (0,1,0)^t$, $\mathbf{e}_3 = (0,0,1)^t$ be the standard basis for $\mathbb{F}^3$. We have that $L(\mathbf{e}_1) = (1,3,0)^t$, $L(\mathbf{e}_2) = (0,4,5)^t$, $L(\mathbf{e}_3) = (2,7,8)^t$. Thus $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$ are columns of the matrix.

Problem. Find a linear mapping $L : \mathbb{F}^3 \to \mathbb{F}^2$ such that $L(\mathbf{e}_1) = (1, 1)$, $L(\mathbf{e}_2) = (0, -2)$, $L(\mathbf{e}_3) = (3, 0)$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{F}^3$.

If such a map exists, then

$$\begin{aligned} L(x, y, z) &= L(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) \\ &= xL(\mathbf{e}_1) + yL(\mathbf{e}_2) + zL(\mathbf{e}_3) \\ &= x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y). \end{aligned}$$

On the other hand, a transformation given by the above formula is indeed linear as

$$L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Notice that columns of the matrix are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$.

Theorem 1 Suppose $L : \mathbb{F}^n \to \mathbb{F}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{F}^n$.

$$\mathbf{y} = A\mathbf{x} \quad \Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Theorem 2 Given $A \in \mathcal{M}_{m,n}(\mathbb{F})$, the matrix of the transformation L_A relative to the standard bases in $\mathbb{F}^n$ and $\mathbb{F}^m$ is exactly A.