MATH 423

Linear Algebra II

Lecture 10:
Inverse matrix.
Change of coordinates.

Let V be a vector space and $\alpha = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be an ordered basis for V.

Theorem 1 The coordinate mapping $C : V \rightarrow \mathbb{F}^n$ given by $C(\mathbf{v}) = [\mathbf{v}]_\alpha$ is linear and invertible (i.e., one-to-one and onto).

Let W be another vector space and $\beta = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ be an ordered basis for W.

Theorem 2 The mapping $M : \mathcal{L}(V, W) \rightarrow \mathcal{M}_{m,n}(\mathbb{F})$ given by $M(L) = [L]_\alpha^\beta$ is linear and invertible.

Linear maps and matrix multiplication

Let V, W, and X be vector spaces. Suppose $\alpha = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is an ordered basis for V, $\beta = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ is an ordered basis for W, and $\gamma = [\mathbf{x}_1, \ldots, \mathbf{x}_k]$ is an ordered basis for X.

Theorem 1 For any linear transformation $L : V \to W$ and any vector $\mathbf{v} \in V$,

$$[L(\mathbf{v})]_\beta = [L]_\alpha^\beta [\mathbf{v}]_\alpha.$$

Theorem 2 For any linear transformations $L : V \to W$ and $T : W \to X$,

$$[T \circ L]_\alpha^\gamma = [T]_\beta^\gamma [L]_\alpha^\beta.$$

Theorem 3 For any linear operators $L : V \to V$ and $T : V \to V$,

$$[T \circ L]_\alpha = [T]_\alpha [L]_\alpha.$$

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is an $n \times n$ matrix $I = (a_{ij})$ such that $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$. It is also denoted I_n.

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$.

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Definition. Let $A \in \mathcal{M}_{n,n}(\mathbb{F})$. Suppose there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Then the matrix A is called **invertible** and B is called the **inverse** of A (denoted A^{-1}).

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Basic properties of inverse matrices:

- If $B = A^{-1}$ then $A = B^{-1}$. In other words, if A is invertible, so is A^{-1}, and $A = (A^{-1})^{-1}$.
- The inverse matrix (if it exists) is unique.
- If $n \times n$ matrices A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
- Similarly, $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$.

Inverting 2×2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$. If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof: Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad - bc)I_2.$$

In the case $\det A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$.
In the case $\det A = 0$, the matrix A is not invertible as otherwise $AB = O \implies A^{-1}(AB) = A^{-1}O = O$
$\implies (A^{-1}A)B = O \implies I_2 B = O \implies B = O$
$\implies A = O$, but the zero matrix is not invertible.

Left multiplication

Any $m \times n$ matrix $A \in \mathcal{M}_{m,n}(\mathbb{F})$ gives rise to a linear transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$ given by $L_A(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in \mathbb{F}^n$ and $L(\mathbf{x}) \in \mathbb{F}^m$ are regarded as column vectors.

Theorem 1 The matrix of the transformation L_A relative to the standard bases in $\mathbb{F}^n$ and $\mathbb{F}^m$ is exactly A.

Theorem 2 Suppose $L : \mathbb{F}^n \to \mathbb{F}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{F}^n$.

Matrix of a linear transformation (revisited)

Let V, W be vector spaces and $f : V \to W$ be a linear map.

Let $\alpha = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ be a basis for V and $g_1 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.

Let $\beta = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ be a basis for W and $g_2 : W \to \mathbb{F}^m$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ g_1 \big\downarrow & & \big\downarrow g_2 \\ \mathbb{F}^n & \longrightarrow & \mathbb{F}^m \end{array}$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{F}^n$ to $\mathbb{F}^m$. It is uniquely represented as $\mathbf{x} \mapsto A\mathbf{x}$, where $A \in \mathcal{M}_{m,n}(\mathbb{F})$.

Theorem $A = [f]_\alpha^\beta$, the matrix of the transformation f relative to the bases α and β.

Change of coordinates

Let V be a vector space of dimension n.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccccc}
 & & V & & \\
 & {\scriptstyle g_1} \swarrow & & \searrow {\scriptstyle g_2} & \\
\mathbb{F}^n & & \longrightarrow & & \mathbb{F}^n
\end{array}
$$

The composition $g_2 \circ g_1^{-1}$ is a linear operator on $\mathbb{F}^n$.
It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \dots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

Problem. Find the transition matrix from the basis $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 2, 1)$ to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

It is convenient to make a two-step transition: first from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and then from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Let U_1 be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and U_2 be the transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \implies$ coordinates $\mathbf{x}$

Basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \implies$ coordinates $U_1\mathbf{x}$

Basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \implies$ coordinates $U_2^{-1}(U_1\mathbf{x}) = (U_2^{-1}U_1)\mathbf{x}$

Thus the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is $U_2^{-1}U_1$.

$$U_2^{-1}U_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

Problem. Consider a linear operator $L : \mathbb{F}^2 \to \mathbb{F}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3,1)$, $\mathbf{v}_2 = (2,1)$.

Let N be the desired matrix. Columns of N are coordinates of the vectors $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ w.r.t. the basis $\mathbf{v}_1, \mathbf{v}_2$.

$$L(\mathbf{v}_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad L(\mathbf{v}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Clearly, $L(\mathbf{v}_2) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2$.

$$L(\mathbf{v}_1) = a\mathbf{v}_1 + b\mathbf{v}_2 \iff \begin{cases} 3a + 2b = 4 \\ a + b = 1 \end{cases} \iff \begin{cases} a = 2 \\ b = -1 \end{cases}$$

Thus $N = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

Change of coordinates for a linear operator

Let $L : V \to V$ be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.

$$
\begin{array}{ccc}
\boxed{\mathbf{a}\text{-coordinates of } \mathbf{v}} & \xrightarrow{A} & \boxed{\mathbf{a}\text{-coordinates of } L(\mathbf{v})} \\
U \big\downarrow & & \big\downarrow U \\
\boxed{\mathbf{b}\text{-coordinates of } \mathbf{v}} & \xrightarrow{B} & \boxed{\mathbf{b}\text{-coordinates of } L(\mathbf{v})}
\end{array}
$$

It follows that $UA\mathbf{x} = BU\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n \implies UA = BU$.

Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

Problem. Consider a linear operator $L : \mathbb{F}^2 \longrightarrow \mathbb{F}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let S be the matrix of L with respect to the standard basis, N be the matrix of L with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$, and U be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{e}_1, \mathbf{e}_2$. Then $N = U^{-1}SU$.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$