MATH 423

Linear Algebra II

# Lecture 11: Change of coordinates (continued). Isomorphism of vector spaces.

# Change of coordinates

Let $V$ be a vector space of dimension $n$.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for $V$ and $g_1 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for $V$ and $g_2 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccccc}
 & & V & & \\
 & {\scriptstyle g_1} \swarrow & & \searrow {\scriptstyle g_2} & \\
\mathbb{F}^n & & \longrightarrow & & \mathbb{F}^n
\end{array}
$$

The composition $g_2 \circ g_1^{-1}$ is a linear operator on $\mathbb{F}^n$.
It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where $U$ is an $n \times n$ matrix.

$U$ is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \dots, \mathbf{u}_n$. Columns of $U$ are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

# Change of coordinates for a linear operator

Let $L : V \to V$ be a linear operator on a vector space $V$.

Let $A$ be the matrix of $L$ relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ for $V$. Let $B$ be the matrix of $L$ relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ for $V$.

Let $U$ be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$.

$$
\begin{array}{ccc}
\boxed{\mathbf{a}\text{-coordinates of } \mathbf{v}} & \xrightarrow{A} & \boxed{\mathbf{a}\text{-coordinates of } L(\mathbf{v})} \\
U \Big\downarrow & & \Big\downarrow U \\
\boxed{\mathbf{b}\text{-coordinates of } \mathbf{v}} & \xrightarrow{B} & \boxed{\mathbf{b}\text{-coordinates of } L(\mathbf{v})}
\end{array}
$$

It follows that $UA\mathbf{x} = BU\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n \implies UA = BU$.
Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

**Problem.** Consider a linear operator $L : \mathbb{F}^2 \to \mathbb{F}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of $L$ with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let $S$ be the matrix of $L$ with respect to the standard basis, $N$ be the matrix of $L$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$, and $U$ be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{e}_1, \mathbf{e}_2$. Then $N = U^{-1}SU$.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

# Similarity

*Definition.* An $n \times n$ matrix $B$ is said to be **similar** to an $n \times n$ matrix $A$ if $\boxed{B = S^{-1}AS}$ for some nonsingular $n \times n$ matrix $S$.

*Remark.* Two $n \times n$ matrices are similar if and only if they represent the same linear operator on $\mathbb{F}^n$ with respect to different bases.

**Theorem** Similarity is an *equivalence relation*, which means that

**(i)** any square matrix $A$ is similar to itself;
**(ii)** if $B$ is similar to $A$, then $A$ is similar to $B$;
**(iii)** if $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

**Theorem** Let $V, W$ be finite-dimensional vector spaces and $f : V \to W$ be a linear map. Then one can choose bases for $V$ and $W$ so that the respective matrix of $f$ is has the block form

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix},$$

where $r$ is the rank of $f$.

*Example.* With a suitable choice of bases, any linear map $f : \mathbb{F}^3 \to \mathbb{F}^2$ has one of the following matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

*Proof of the theorem:*

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a basis for the null-space $\mathcal{N}(f)$.

Extend it to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_1, \ldots, \mathbf{u}_r$ for $V$.

Then $f(\mathbf{u}_1), f(\mathbf{u}_2), \ldots, f(\mathbf{u}_r)$ is a basis for the range $\mathcal{R}(f)$.

Extend it to a basis $f(\mathbf{u}_1), \ldots, f(\mathbf{u}_r), \mathbf{w}_1, \ldots, \mathbf{w}_l$ for $W$.

Now the matrix of $f$ with respect to bases $[\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_k]$ and $[f(\mathbf{u}_1), \ldots, f(\mathbf{u}_r), \mathbf{w}_1, \ldots, \mathbf{w}_l]$ is

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}.$$

*Definition.* A map $f : V_1 \to V_2$ is **one-to-one** if it maps different elements from $V_1$ to different elements in $V_2$. The map $f$ is **onto** if any element $\mathbf{y} \in V_2$ is represented as $f(\mathbf{x})$ for some $\mathbf{x} \in V_1$.

If the map $f$ is both one-to-one and onto, then the inverse map $f^{-1} : V_2 \to V_1$ is well defined.

Now let $V_1, V_2$ be vector spaces and $L : V_1 \to V_2$ be a linear map.

**Theorem** **(i)** The linear map $L$ is one-to-one if and only if $\mathcal{N}(L) = \{\mathbf{0}\}$.
**(ii)** The linear map $L$ is onto if $\mathcal{R}(L) = V_2$.
**(iii)** If the linear map $L$ is both one-to-one and onto, then the inverse map $L^{-1}$ is also linear.

# Isomorphism

*Definition.* A linear map $L : V_1 \to V_2$ is called an **isomorphism** of vector spaces if it is both one-to-one and onto.

The vector space $V_1$ is said to be **isomorphic** to $V_2$ if there exists an isomorphism $L : V_1 \to V_2$.

*The word "isomorphism" applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where "corresponding" means that the two parts play similar roles in their respective structures.*

# Alternative notation

*General maps*

| | |
|---|---|
| one-to-one | injective |
| onto | surjective |
| one-to-one and onto | bijective |

*Linear maps*

| | |
|---|---|
| any map | homomorphism |
| one-to-one | monomorphism |
| onto | epimorphism |
| one-to-one and onto | isomorphism |

*Linear self-maps*

| | |
|---|---|
| any map | endomorphism |
| one-to-one and onto | automorphism |

# Examples of isomorphism

- $\mathcal{M}_{1,3}(\mathbb{F})$ is isomorphic to $\mathcal{M}_{3,1}(\mathbb{F})$.

Isomorphism: $(x_1, x_2, x_3) \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

- $\mathcal{M}_{2,2}(\mathbb{F})$ is isomorphic to $\mathbb{F}^4$.

Isomorphism: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$.

- $\mathcal{M}_{2,3}(\mathbb{F})$ is isomorphic to $\mathcal{M}_{3,2}(\mathbb{F})$.

Isomorphism: $A \mapsto A^t$.

- The plane $z = 0$ in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

Isomorphism: $(x, y, 0) \mapsto (x, y)$.

# Examples of isomorphism

- $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$.

Isomorphism: $a_0+a_1x+\cdots+a_nx^n \mapsto (a_0, a_1, \ldots, a_n)$.

- $\mathcal{P}$ is isomorphic to $\mathbb{R}_0^\infty$.

Isomorphism:
$a_0 + a_1x + \cdots + a_nx^n \mapsto (a_0, a_1, \ldots, a_n, 0, 0, \ldots)$.

- $\mathcal{M}_{m,n}(\mathbb{F})$ is isomorphic to $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.

Isomorphism: $A \mapsto L_A$, where $L_A(\mathbf{x}) = A\mathbf{x}$.

- Any vector space $V$ of dimension $n$ is isomorphic to $\mathbb{F}^n$.

Isomorphism: $\mathbf{v} \mapsto [\mathbf{v}]_\alpha$, where $\alpha$ is a basis for $V$.

## Isomorphism and dimension

*Definition.* Two sets $S_1$ and $S_2$ are said to be of the same **cardinality** if there exists a bijective map $f : S_1 \to S_2$.

**Theorem 1** All bases of a fixed vector space $V$ are of the same cardinality.

**Theorem 2** Two vector spaces are isomorphic if and only if their bases are of the same cardinality. In particular, a vector space $V$ is isomorphic to $\mathbb{F}^n$ if and only if $\dim V = n$.

*Remark.* For a finite set, the cardinality is a synonym for the number of its elements. For an infinite set, the cardinality is a more sophisticated notion. For example, $\mathbb{R}^\infty$ and $\mathcal{P}$ are both infinite-dimensional vector spaces but they are not isomorphic.