MATH 423

Linear Algebra II

Lecture 12: Review for Test 1.

Topics for Test 1

Vector spaces (F/I/S 1.1–1.7, 2.2, 2.4)

- Vector spaces: axioms and basic properties.
- Basic examples of vector spaces (coordinate vectors, matrices, polynomials, functional spaces).
- Subspaces.
- Span, spanning set.
- Linear independence.
- Basis and dimension.
- Various characterizations of a basis.
- Basis and coordinates.
- Change of coordinates, transition matrix.
- * Vector space over a field.

Topics for Test 1

Linear transformations (F/I/S 2.1–2.5)

- Linear transformations: definition and basic properties.
- Linear transformations: basic examples.
- Vector space of linear transformations.
- Range and null-space of a linear map.
- Matrix of a linear transformation.
- Matrix algebra and composition of linear maps.
- Characterization of linear maps from $\mathbb{F}^n$ to $\mathbb{F}^m$.
- Change of coordinates for a linear operator.
- Isomorphism of vector spaces.

Sample problems for Test 1

Problem 1 (20 pts.) Let $\mathcal{P}_3$ be the vector space of all polynomials (with real coefficients) of degree at most 3. Determine which of the following subsets of $\mathcal{P}_3$ are subspaces. Briefly explain.

(i) The set S_1 of polynomials $p(x) \in \mathcal{P}_3$ such that $p(0) = 0$.

(ii) The set S_2 of polynomials $p(x) \in \mathcal{P}_3$ such that $p(0) = 0$ and $p(1) = 0$.

(iii) The set S_3 of polynomials $p(x) \in \mathcal{P}_3$ such that $p(0) = 0$ or $p(1) = 0$.

(iv) The set S_4 of polynomials $p(x) \in \mathcal{P}_3$ such that $(p(0))^2 + 2(p(1))^2 + (p(2))^2 = 0$.

Sample problems for Test 1

Problem 2 (20 pts.) Let V be a subspace of $\mathcal{F}(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x}),\ \sinh x = \frac{1}{2}(e^x - e^{-x})$.

Problem 3 (25 pts.) Suppose V_1 and V_2 are subspaces of a vector space V such that $\dim V_1 = 5,\ \dim V_2 = 3,\ \dim(V_1 + V_2) = 6$. Find $\dim(V_1 \cap V_2)$. Explain your answer.

Sample problems for Test 1

Problem 4 (25 pts.) Consider a linear transformation $T : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,3}(\mathbb{R})$ given by

$$T(A) = A \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ for all } 2 \times 2 \text{ matrices } A.$$

Find bases for the range and for the null-space of T.

Bonus Problem 5 (15 pts.) Suppose V_1 and V_2 are real vector spaces, $\dim V_1 = m$, $\dim V_2 = n$. Let $B(V_1, V_2)$ denote the subspace of $\mathcal{F}(V_1 \times V_2)$ consisting of bilinear functions (i.e., functions of two variables $\mathbf{x} \in V_1$ and $\mathbf{y} \in V_2$ that depend linearly on each variable). Prove that $B(V_1, V_2)$ is isomorphic to $\mathcal{M}_{m,n}(\mathbb{R})$.

Problem 1. Let $\mathcal{P}_3$ be the vector space of all polynomials (with real coefficients) of degree at most 3. Determine which of the following subsets of $\mathcal{P}_3$ are vector subspaces. Briefly explain.

How to check whether a subset S of a vector space is a subspace?

- Default approach: show that S is a nonempty set closed under addition and scalar multiplication.
- Represent S as the span of some collection of vectors.
- Represent S as the null-space of a linear transformation.
- Represent S as the intersection of some known subspaces.

(i) The set S_1 of polynomials $p(x) \in \mathcal{P}_3$ such that $p(0) = 0$.

S_1 is not empty because it contains the zero polynomial.

$p_1(0) = p_2(0) = 0 \implies p_1(0) + p_2(0) = 0$
$\implies (p_1 + p_2)(0) = 0.$
Hence S_1 is closed under addition.

$p(0) = 0 \implies rp(0) = 0 \implies (rp)(0) = 0.$
Hence S_1 is closed under scalar multiplication.

Thus S_1 is a subspace of $\mathcal{P}_3$.

(i) The set S_1 of polynomials $p(x) \in \mathcal{P}_3$ such that $p(0) = 0$.

Alternatively, consider a functional $\ell : \mathcal{P}_3 \to \mathbb{R}$ given by $\ell[p(x)] = p(0)$.

It is easy to see that ℓ is a linear functional.

Clearly, S_1 is the null-space of ℓ, hence it is a subspace of $\mathcal{P}_3$.

(ii) The set S_2 of polynomials $p(x) \in \mathcal{P}_3$ such that $p(0) = 0$ and $p(1) = 0$.

- S_2 contains the zero polynomial,
- S_2 is closed under addition,
- S_2 is closed under scalar multiplication.

Thus S_2 is a subspace of $\mathcal{P}_3$.

Alternatively, let S_1' denote the set of polynomials $p(x) \in \mathcal{P}_3$ such that $p(1) = 0$. The set S_1' is a subspace of $\mathcal{P}_3$ for the same reason as S_1. Clearly, $S_2 = S_1 \cap S_1'$. Now the intersection of two subspaces of $\mathcal{P}_3$ is also a subspace.

Alternatively, S_2 is the null-space of a linear transformation $L : \mathcal{P}_3 \to \mathbb{R}^2$ given by $L[p(x)] = (p(0), p(1))$.

(iii) The set S_3 of polynomials $p(x) \in \mathcal{P}_3$ such that $p(0) = 0$ or $p(1) = 0$.

- S_3 contains the zero polynomial,
- S_3 is closed under scalar multiplication,
- however S_3 is not closed under addition.

For example, $p_1(x) = x$ and $p_2(x) = x - 1$ are in S_3 but $(p_1 + p_2)(x) = 2x - 1$ is not in S_3.

Thus S_3 is **not** a subspace of $\mathcal{P}_3$.

(iv) The set S_4 of polynomials $p(x) \in \mathcal{P}_3$ such that $(p(0))^2 + 2(p(1))^2 + (p(2))^2 = 0$.

Since coefficients of a polynomial $p(x) \in \mathcal{P}_3$ are real, it belongs to S_4 if and only if $p(0) = p(1) = p(2) = 0$.

Hence S_4 is the null-space of a linear transformation $L : \mathcal{P}_3 \to \mathbb{R}^3$ given by $L[p(x)] = (p(0), p(1), p(2))$. Thus S_4 is a subspace.

Problem 2. Let V be a subspace of $\mathcal{F}(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Let α denote the basis e^x, e^{-x} and β denote the basis $\cosh x$, $\sinh x$ for V. Let A denote the matrix of the operator L relative to α (which is given) and B denote the matrix of L relative to β (which is to be found). By definition of the functions $\cosh x$ and $\sinh x$, the transition matrix from β to α is $U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. It follows that $B = U^{-1}AU$. One easily checks that $2U^2 = I$. Hence $U^{-1} = 2U$ so that

$$B = U^{-1}AU = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}.$$

Problem 3. Suppose V_1 and V_2 are subspaces of a vector space V such that $\dim V_1 = 5$, $\dim V_2 = 3$, $\dim(V_1 + V_2) = 6$. Find $\dim(V_1 \cap V_2)$. Explain your answer.

We are going to show that $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 + V_2)$ for any finite-dimensional subspaces V_1 and V_2. In our particular case this will imply that $\dim(V_1 \cap V_2) = 2$.

First we choose a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ for the intersection $V_1 \cap V_2$. The set $S_0 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent in both V_1 and V_2. Therefore we can extend this set to a basis for V_1 and to a basis for V_2. Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ be vectors that extend S_0 to a basis for V_1 and $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ be vectors that extend S_0 to a basis for V_2. It remains to show that $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_n$ is a basis for $V_1 + V_2$. Then $\dim V_1 = k + m$, $\dim V_2 = k + n$, $\dim(V_1 + V_2) = k + m + n$, and $\dim(V_1 \cap V_2) = k$.

By definition, the subspace $V_1 + V_2$ consists of vector sums $\mathbf{x} + \mathbf{y}$, where $\mathbf{x} \in V_1$ and $\mathbf{y} \in V_2$. Since $\mathbf{x}$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m$ and $\mathbf{y}$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \dots, \mathbf{w}_n$, it follows that $\mathbf{x} + \mathbf{y}$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m$, $\mathbf{w}_1, \dots, \mathbf{w}_n$. Therefore these vectors span $V_1 + V_2$.

Now we prove that vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_n$ are linearly independent. Assume

$$r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k + s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m + t_1\mathbf{w}_1 + \cdots + t_n\mathbf{w}_n = \mathbf{0}$$

for some scalars r_i, s_j, t_l. Let $\mathbf{x} = s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m$, $\mathbf{y} = t_1\mathbf{w}_1 + \cdots + t_n\mathbf{w}_n$, and $\mathbf{z} = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$. Then $\mathbf{x} \in V_1$, $\mathbf{y} \in V_2$, and $\mathbf{z} \in V_1 \cap V_2$. The equality $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$ implies that $\mathbf{x} = -\mathbf{y} - \mathbf{z} \in V_2$ and $\mathbf{y} = -\mathbf{x} - \mathbf{z} \in V_1$. Hence both $\mathbf{x}$ and $\mathbf{y}$ are in $V_1 \cap V_2$.

From this we derive that $\mathbf{x} = \mathbf{y} = \mathbf{z} = \mathbf{0}$. It follows that all coefficients are zeros. Thus the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_n$ are linearly independent.

Problem 4. Consider a linear transformation $T : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,3}(\mathbb{R})$ given by

$$T(A) = A \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

for all 2×2 matrices A. Find bases for the range and for the null-space of T.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $T(A) = \begin{pmatrix} a+b & a & a \\ c+d & c & c \end{pmatrix}$
$= aB_1 + bB_2 + cB_3 + dB_4$, where $B_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B_2 =$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $B_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.
Therefore the range of T is spanned by the matrices B_1, B_2, B_3, B_4. If $aB_1 + bB_2 + cB_3 + dB_4 = O$ for some scalars $a, b, c, d \in \mathbb{R}$, then $a + b = a = c + d = d = 0$, which implies $a = b = c = d = 0$. Therefore B_1, B_2, B_3, B_4 are linearly independent so that they form a basis for the range of T. Also, it follows that the null-space of T is trivial.

Bonus Problem 5. Suppose V_1 and V_2 are real vector spaces of dimension m and n, respectively. Let $B(V_1, V_2)$ denote the subspace of $\mathcal{F}(V_1 \times V_2)$ consisting of bilinear functions (i.e., functions of two variables $x \in V_1$ and $y \in V_2$ that depend linearly on each variable). Prove that $B(V_1, V_2)$ is isomorphic to $\mathcal{M}_{m,n}(\mathbb{R})$.

Let $\alpha = [\mathbf{v}_1, \dots, \mathbf{v}_m]$ be an ordered basis for V_1 and $\beta = [\mathbf{w}_1, \dots, \mathbf{w}_n]$ be an ordered basis for V_2. For any matrix $C \in \mathcal{M}_{m,n}(\mathbb{R})$ we define a function $f_C : V_1 \times V_2 \to \mathbb{R}$ by $f_C(\mathbf{x}, \mathbf{y}) = ([\mathbf{x}]_\alpha)^t C [\mathbf{y}]_\beta$ for all $\mathbf{x} \in V_1$ and $\mathbf{y} \in V_2$.

It is easy to observe that f_C is bilinear. Moreover, the expression $f_C(\mathbf{x}, \mathbf{y})$ depends linearly on C as well. This implies that a transformation $L : \mathcal{M}_{m,n}(\mathbb{R}) \to B(V_1, V_2)$ given by $L(C) = f_C$ is linear. The transformation L is one-to-one since the matrix C can be recovered from the function f_C. Namely, if $C = (c_{ij})$, then $c_{ij} = f_C(\mathbf{v}_i, \mathbf{w}_j)$, $1 \le i \le m$, $1 \le j \le n$.

It remains to show that L is onto. Take any function $f \in B(V_1, V_2)$ and vectors $\mathbf{x} \in V_1$, $\mathbf{y} \in V_2$. We have $\mathbf{x} = r_1\mathbf{v}_1 + \cdots + r_m\mathbf{v}_m$ and $\mathbf{y} = s_1\mathbf{w}_1 + \cdots + s_n\mathbf{w}_n$ for some scalars r_i, s_j. Using bilinearity of f, we obtain

$$f(\mathbf{x}, \mathbf{y}) = f(r_1\mathbf{v}_1 + \cdots + r_m\mathbf{v}_m, \mathbf{y}) = \sum_{i=1}^{m} r_i\, f(\mathbf{v}_i, \mathbf{y})$$

$$= \sum_{i=1}^{m} r_i\, f(\mathbf{v}_i, s_1\mathbf{w}_1 + \cdots + s_n\mathbf{w}_n) = \sum_{i=1}^{m}\sum_{j=1}^{n} r_i s_j\, f(\mathbf{v}_i, \mathbf{w}_j)$$

$$= (r_1, r_2 \ldots, r_m) \begin{pmatrix} f(\mathbf{v}_1, \mathbf{w}_1) & f(\mathbf{v}_1, \mathbf{w}_2) & \ldots & f(\mathbf{v}_1, \mathbf{w}_n) \\ f(\mathbf{v}_2, \mathbf{w}_1) & f(\mathbf{v}_2, \mathbf{w}_2) & \ldots & f(\mathbf{v}_2, \mathbf{w}_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(\mathbf{v}_m, \mathbf{w}_1) & f(\mathbf{v}_m, \mathbf{w}_2) & \ldots & f(\mathbf{v}_m, \mathbf{w}_n) \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$$

$$= ([\mathbf{x}]_\alpha)^t\, C\, [\mathbf{y}]_\beta$$

so that $f = f_C$ for some matrix $C \in \mathcal{M}_{m,n}(\mathbb{R})$.