MATH 423

Linear Algebra II

Lecture 13:
Advanced constructions of vector spaces.

Cartesian product

Given two sets V_1 and V_2, the **Cartesian product** $V_1 \times V_2$ is the set of all pairs $(\mathbf{x}, \mathbf{y})$, where $\mathbf{x} \in V_1$ and $\mathbf{y} \in V_2$.

If both V_1 and V_2 are vector spaces (over the same field $\mathbb{F}$) then $V_1 \times V_2$ is naturally endowed with the structure of a vector space. Namely, the linear operations are given by

$$(\mathbf{x}_1, \mathbf{y}_1) + (\mathbf{x}_2, \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2),$$
$$r(\mathbf{x}, \mathbf{y}) = (r\mathbf{x}, r\mathbf{y})$$

for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x} \in V_1$, $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y} \in V_2$, and $r \in \mathbb{F}$.

Note that the zero vector in $V_1 \times V_2$ is $(\mathbf{0}_1, \mathbf{0}_2)$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are the zero vectors in V_1 and V_2, respectively.

Theorem $\dim(V_1 \times V_2) = \dim V_1 + \dim V_2$.

The theorem follows from the next lemma.

Lemma Suppose S_1 is a basis for V_1 and S_2 is a basis for V_2. Then the union of sets $S_1 \times \{\mathbf{0}_2\}$ and $\{\mathbf{0}_1\} \times S_2$ is a basis for $V_1 \times V_2$.

Idea of the proof: $(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{0}_2) + (\mathbf{0}_1, \mathbf{y})$ for all $\mathbf{x} \in V_1$, $\mathbf{y} \in V_2$. Also, if $\mathbf{x}_1, \dots, \mathbf{x}_m \in S_1$, $\mathbf{y}_1, \dots, \mathbf{y}_n \in S_2$, then

$$r_1(\mathbf{x}_1, \mathbf{0}_2) + \cdots + r_m(\mathbf{x}_m, \mathbf{0}_2) + s_1(\mathbf{0}_1, \mathbf{y}_1) + \cdots + s_n(\mathbf{0}_1, \mathbf{y}_n)$$
$$= (r_1\mathbf{x}_1 + \cdots + r_m\mathbf{x}_m, s_1\mathbf{y}_1 + \dots, s_n\mathbf{y}_n).$$

Similarly, for any vector spaces $V_1, V_2, \dots, V_k$ we can define a vector space $V_1 \times V_2 \times \cdots \times V_k$. The dimension of this space is $\sum_{i=1}^{k} \dim V_i$.

Example. $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$.

Direct sum

Let V be a vector space. For any subsets $X_1, X_2, \dots, X_n$ of V we define another subset

$$X_1 + X_2 + \cdots + X_n = \{\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n \mid \mathbf{x}_i \in X_i,\ 1 \leq i \leq n\}.$$

Theorem The set $X_1 + X_2 + \cdots + X_n$ is a subspace of V provided that each X_i is a subspace of V.

Suppose $V = V_1 + V_2 + \cdots + V_n$ for some subspaces $V_1, \dots, V_n$. We say that V is the **direct sum** of the subspaces V_i and write $\boxed{V = V_1 \oplus V_2 \oplus \cdots \oplus V_n}$ if any vector $\mathbf{x} \in V$ is uniquely expanded as $\mathbf{x}_1 + \cdots + \mathbf{x}_n$, where $\mathbf{x}_i \in V_i$.

Example. $V_1 \times V_2 = (V_1 \times \{\mathbf{0}_2\}) \oplus (\{\mathbf{0}_1\} \times V_2)$ for any vector spaces V_1 and V_2. The expansion is

$$(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{0}_2) + (\mathbf{0}_1, \mathbf{y}).$$

Suppose $V_1, V_2, \ldots, V_n$ are subspaces of a vector space V. Consider a mapping $f : V_1 \times \cdots \times V_n \to V$ given by

$$f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n.$$

Theorem 1 **(i)** The mapping f is linear.
(ii) $V = V_1 + V_2 + \cdots + V_n$ if and only if f is onto.
(iii) $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ if and only if f is an isomorphism.

Corollary $\dim(V_1 \oplus V_2 \oplus \cdots \oplus V_n) = \sum_{i=1}^{n} \dim V_i$.

Theorem 2 Suppose V_1 and V_2 are subspaces of V. Then the sum $V_1 + V_2$ is direct if and only if $V_1 \cap V_2 = \{\mathbf{0}\}$.

Linear operations on sets

Let V be a vector space. Given two nonempty subsets X and Y of V, we define another subset, denoted $X+Y$, by $X+Y = \{\mathbf{x}+\mathbf{y} \mid \mathbf{x} \in X,\ \mathbf{y} \in Y\}$.
Given a nonempty subset $X \subset V$ and a scalar $r \in \mathbb{F}$, we define another subset, denoted rX, by $rX = \{r\mathbf{x} \mid \mathbf{x} \in X\}$.

The set of all nonempty subsets of V is **not** a vector space with respect to these operations unless $V = \{\mathbf{0}\}$.

Indeed, we have $X + \{\mathbf{0}\} = X$ and $X + V = V$ for any nonempty subset $X \subset V$. The first relation implies that only $\{\mathbf{0}\}$ could be the zero vector. Then the second relation implies that the set V has no additive inverse so that the axiom VS4 fails.

Quotient space

Let V_0 be a subspace of a vector space V. A **coset** of V_0 in V is any set of the form $\{\mathbf{x}\} + V_0$ (also denoted $\mathbf{x} + V_0$). The set of all cosets of V_0 is denoted V/V_0 and called the **quotient** of V by V_0.

Theorem 1 V/V_0 is a vector space.

The theorem follows from the next lemma.

Lemma $(\mathbf{x} + V_0) + (\mathbf{y} + V_0) = (\mathbf{x} + \mathbf{y}) + V_0$ and $r(\mathbf{x} + V_0) = r\mathbf{x} + V_0$ for any vectors $\mathbf{x}, \mathbf{y} \in V$ and scalar r.

Theorem 2 $\dim(V/V_0) = \dim V - \dim V_0$.

Proof: Consider a mapping $\phi : V \to V/V_0$ given by $\phi(\mathbf{x}) = \mathbf{x} + V_0$ for all $\mathbf{x} \in V$. By the above lemma, ϕ is linear. Clearly, ϕ is onto so that the range of ϕ is V/V_0. The zero vector of the vector space V/V_0 is $\mathbf{0} + V_0 = V_0$. It follows that the null-space of ϕ is V_0. By the dimension theorem, $\dim(V/V_0) + \dim V_0 = \dim V$.

Given vector spaces V_1 and V_2, let $B(V_1, V_2)$ denote the subspace of $\mathcal{F}(V_1 \times V_2, \mathbb{F})$ consisting of **bilinear functions** (i.e., functions of two variables $\mathbf{x} \in V_1$ and $\mathbf{y} \in V_2$ that depend linearly on each variable).

Theorem If $\dim V_1 = m$ and $\dim V_2 = n$, then $B(V_1, V_2)$ is isomorphic to $\mathcal{M}_{m,n}(\mathbb{F})$.

Proof: Let $\alpha = [\mathbf{v}_1, \dots, \mathbf{v}_m]$ be an ordered basis for V_1 and $\beta = [\mathbf{w}_1, \dots, \mathbf{w}_n]$ be an ordered basis for V_2. For any matrix $C \in \mathcal{M}_{m,n}(\mathbb{F})$ we define a function $f_C : V_1 \times V_2 \to \mathbb{R}$ by $f_C(\mathbf{x}, \mathbf{y}) = ([\mathbf{x}]_\alpha)^t C [\mathbf{y}]_\beta$ for all $\mathbf{x} \in V_1$ and $\mathbf{y} \in V_2$.

It is easy to observe that f_C is bilinear. Moreover, the expression $f_C(\mathbf{x}, \mathbf{y})$ depends linearly on C as well. This implies that a transformation $L : \mathcal{M}_{m,n}(\mathbb{F}) \to B(V_1, V_2)$ given by $L(C) = f_C$ is linear. The transformation L is one-to-one since the matrix C can be recovered from the function f_C. Namely, if $C = (c_{ij})$, then $c_{ij} = f_C(\mathbf{v}_i, \mathbf{w}_j)$, $1 \leq i \leq m$, $1 \leq j \leq n$.

It remains to show that L is onto. Take any function $f \in B(V_1, V_2)$ and vectors $\mathbf{x} \in V_1$, $\mathbf{y} \in V_2$. We have $\mathbf{x} = r_1\mathbf{v}_1 + \cdots + r_m\mathbf{v}_m$ and $\mathbf{y} = s_1\mathbf{w}_1 + \cdots + s_n\mathbf{w}_n$ for some scalars r_i, s_j. Using bilinearity of f, we obtain

$$f(\mathbf{x}, \mathbf{y}) = f(r_1\mathbf{v}_1 + \cdots + r_m\mathbf{v}_m, \mathbf{y}) = \sum_{i=1}^{m} r_i\, f(\mathbf{v}_i, \mathbf{y})$$

$$= \sum_{i=1}^{m} r_i\, f(\mathbf{v}_i, s_1\mathbf{w}_1 + \cdots + s_n\mathbf{w}_n) = \sum_{i=1}^{m}\sum_{j=1}^{n} r_i s_j\, f(\mathbf{v}_i, \mathbf{w}_j)$$

$$= (r_1, r_2 \dots, r_m) \begin{pmatrix} f(\mathbf{v}_1, \mathbf{w}_1) & f(\mathbf{v}_1, \mathbf{w}_2) & \dots & f(\mathbf{v}_1, \mathbf{w}_n) \\ f(\mathbf{v}_2, \mathbf{w}_1) & f(\mathbf{v}_2, \mathbf{w}_2) & \dots & f(\mathbf{v}_2, \mathbf{w}_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(\mathbf{v}_m, \mathbf{w}_1) & f(\mathbf{v}_m, \mathbf{w}_2) & \dots & f(\mathbf{v}_m, \mathbf{w}_n) \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$$

$$= ([\mathbf{x}]_\alpha)^t\, C\, [\mathbf{y}]_\beta$$

for some matrix $C \in \mathcal{M}_{m,n}(\mathbb{F})$. Then $f = f_C$.