MATH 423

Linear Algebra II

Lecture 14:
General linear equations.
Elementary matrices.

General linear equations

Definition. A **linear equation** is an equation of the form

$$\boxed{L(\mathbf{x}) = \mathbf{b},}$$

where $L : V \to W$ is a linear mapping, $\mathbf{b}$ is a given vector from W, and $\mathbf{x}$ is an unknown vector from V.

The range of L is the set of all vectors $\mathbf{b} \in W$ such that the equation $L(\mathbf{x}) = \mathbf{b}$ has a solution.

The null-space of L is the solution set of the **homogeneous** linear equation $L(\mathbf{x}) = \mathbf{0}$.

Theorem If the linear equation $L(\mathbf{x}) = \mathbf{b}$ is solvable and $\dim \mathcal{N}(L) < \infty$, then the general solution is

$$\mathbf{x}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k,$$

where $\mathbf{x}_0$ is a particular solution, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for the null-space $\mathcal{N}(L)$, and $t_1, \ldots, t_k$ are arbitrary scalars.

Example. $\begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases}$

$$L : \mathbb{R}^3 \to \mathbb{R}^2, \quad L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Linear equation: $L(\mathbf{x}) = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

$$\begin{cases} x + y + z = 4 \\ x + 2y = 3 \end{cases} \iff \begin{cases} x + y + z = 4 \\ y - z = -1 \end{cases}$$

$$\iff \begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases}$$

$$(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).$$

Example. $u'''(x) - 2u''(x) + u'(x) = e^{2x}$.

Linear operator $L : C^3(\mathbb{R}) \to C(\mathbb{R})$, $Lu = u''' - 2u'' + u'$.
Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

According to the theory of differential equations, the initial value problem

$$\begin{cases} u'''(x) - 2u''(x) + u'(x) = g(x), \quad x \in \mathbb{R}, \\ u(a) = b_0, \\ u'(a) = b_1, \\ u''(a) = b_2 \end{cases}$$

has a unique solution for any $g \in C(\mathbb{R})$ and any $b_0, b_1, b_2 \in \mathbb{R}$. It follows that $L(C^3(\mathbb{R})) = C(\mathbb{R})$.

Also, the initial data evaluation $I(u) = (u(a), u'(a), u''(a))$, which is a linear mapping $I : C^3(\mathbb{R}) \to \mathbb{R}^3$, is one-to-one and onto when restricted to $\mathcal{N}(L)$. Hence $\dim \mathcal{N}(L) = 3$.

It is easy to check that $L(xe^x) = L(e^x) = L(1) = 0$. One can also show that xe^x, e^x, and 1 are linearly independent.

Example. $u'''(x) - 2u''(x) + u'(x) = e^{2x}$.

Linear operator $L : C^3(\mathbb{R}) \to C(\mathbb{R})$,
$Lu = u''' - 2u'' + u'$.

Linear equation: $Lu = b$, where $b(x) = e^{2x}$.

It follows from the previous slide that functions xe^x, e^x and 1 form a basis for the null-space of L. It remains to find a particular solution.

$L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}$.
Since L is a linear operator, $L\left(\frac{1}{2}e^{2x}\right) = e^{2x}$.

Particular solution: $u_0(x) = \frac{1}{2}e^{2x}$.

Thus the general solution is

$$u(x) = \frac{1}{2}e^{2x} + t_1 xe^x + t_2 e^x + t_3.$$

Elementary row operations for matrices:

(1) to interchange two rows;

(2) to multiply a row by a nonzero scalar;

(3) to add the ith row multiplied by some scalar r to the jth row.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Similarly, we define three types of **elementary column operations**.

Elementary row operations

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \hline a_{21} & a_{22} & \dots & a_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix},$$

where $\mathbf{v}_i = (a_{i1} \; a_{i2} \; \dots \; a_{in})$ is a row vector.

Elementary row operations

Operation 1: to interchange the ith row with the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 2: to multiply the ith row by $r \neq 0$:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 3: to add the ith row multiplied by r to the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \to \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Theorem Any elementary row operation can be simulated as left multiplication by a certain matrix.

Examples.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 2b_1 & 2b_2 & 2b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+3a_1 & b_2+3a_2 & b_3+3a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & O & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & O & & & & & 1 \end{pmatrix} \quad \begin{matrix} \\ \\ \text{row } \#i \\ \\ \text{row } \#j \\ \\ \\ \end{matrix}$$

To obtain the matrix EA from A, interchange the ith row with the jth row. To obtain AE from A, interchange the ith column with the jth column.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & O & \\ & & 1 & & & & \\ & & & r & & & \\ & & & & 1 & & \\ & O & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad \text{row } \#i$$

To obtain the matrix EA from A, multiply the ith row by r. To obtain the matrix AE from A, multiply the ith column by r.

Elementary matrices

$$E = \begin{pmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & O & \\ 0 & \cdots & 1 & & & & \\ \vdots & & \vdots & \ddots & & & \\ 0 & \cdots & r & \cdots & 1 & & \\ \vdots & & \vdots & & \vdots & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \begin{matrix} \\ \\ \text{row } \#i \\ \\ \text{row } \#j \\ \\ \\ \end{matrix}$$

To obtain the matrix EA from A, add r times the ith row to the jth row. To obtain the matrix AE from A, add r times the jth column to the ith column.

Notice that the elementary matrix E_σ simulating an elementary row operation σ is obtained by applying σ to the identity matrix. In particular, this implies that E_σ is unique.

Theorem Any elementary row operation σ_1 can be undone by applying another elementary row operation σ_2. Moreover, the operation σ_1 will undo the operation σ_2.

Corollary Elementary matrices are invertible.

Proof: Let E be an elementary matrix simulating an elementary row operation σ. Let τ be the operation such that σ and τ undo each other. The operation τ is simulated as left multiplication by some matrix E_0. Then $E_0EA = EE_0A = A$ for any matrix A. When $A = I$, we get $E_0E = EE_0 = I$.