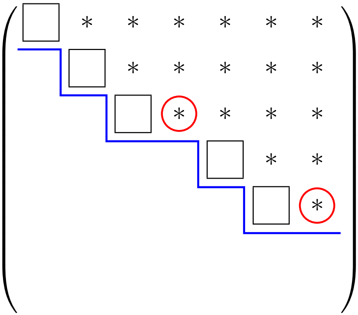
MATH 423

Linear Algebra II

Lecture 17:
Reduced row echelon form (continued).
Determinant of a matrix.

Row echelon form

A matrix is said to be in the **row echelon form** if the leading entries shift to the right as we go from the first row to the last one.



- Leading entries are boxed;
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry.

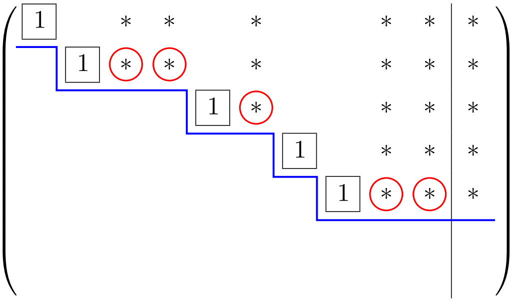
Strict triangular form is a particular case of row echelon form that can occur only for square matrices:

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

- no zero rows;
- there is a leading entry in each column.

Reduced row echelon form

A matrix is said to be in the **reduced row echelon form** if
(i) it is in the row echelon form (i.e., leading entries shift to the right as we go from the first row to the last one);
(ii) each leading entry is equal to 1;
(iii) each leading entry is the only nonzero entry in its column.



- All entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are 0.

Theorem Any matrix can be converted into row echelon form by applying elementary row operations.

Sketch of the proof: The proof is by induction on the number of columns in the matrix. It relies on the next lemma.

Lemma Any matrix A can be converted to one of the following forms using elementary row operations:

(i) O (the zero matrix); **(ii)** $(1\ a_{12}\ a_{13}\ \dots\ a_{1n})$;

(iii) $\left(\begin{array}{c|c} 0 & \\ \vdots & B \\ 0 & \end{array}\right)$; **(iv)** $\left(\begin{array}{c|ccc} 1 & a_{12} & \dots & a_{1n} \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right)$.

In the cases (i) and (ii), we already have a row echelon form. In the cases (iii) and (iv), it is enough to convert the matrix B to row echelon form. Moreover, the row reduction on the block B can be done by applying elementary row operations to the entire matrix.

Theorem Any matrix in row echelon form can be converted into reduced row echelon form by applying elementary row operations.

Example. $$A = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & 0 & a_{16} \\ 0 & 1 & a_{23} & a_{24} & 0 & a_{26} \\ 0 & 0 & 0 & a_{34} & 0 & a_{36} \\ 0 & 0 & 0 & 0 & 1 & a_{46} \end{pmatrix}, \quad a_{11}, a_{34} \neq 0.$$

The matrix A is in row echelon form. Columns #2 and #5 are RREF ready, columns #1 and #4 are not. To prepare column #4 for RREF, we multiply row #3 by a_{34}^{-1}, then subtract row #3 times a_{24} from row #2 and subtract row #3 times a_{14} from row #1:

$$A' = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 & 0 & a'_{16} \\ 0 & 1 & a_{23} & 0 & 0 & a'_{26} \\ 0 & 0 & 0 & 1 & 0 & a'_{36} \\ 0 & 0 & 0 & 0 & 1 & a_{46} \end{pmatrix}.$$

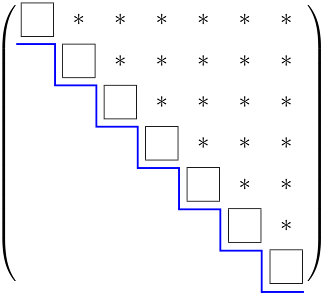
Let C be a matrix in the row echelon form (resp. reduced row echelon form). We say that C is a **row echelon form** (resp. **reduced row echelon form**) of a matrix A if C can be obtained from A by applying elementary row operations.

Theorem If a matrix C is in row echelon form, then the nonzero rows of C are linearly independent.

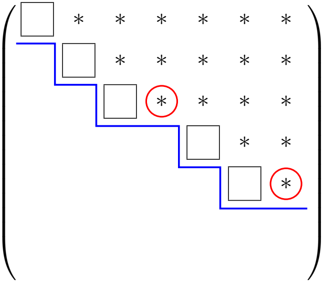
Corollary 1 The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

Corollary 2 If a square matrix A is invertible then its row echelon form is also in strict triangular form. Otherwise the row echelon form of A contains a zero row.

Row echelon form of a square matrix:



invertible case



noninvertible case

Characterizations of invertible matrices

Theorem Given an $n \times n$ matrix A, the following conditions are equivalent:

(i) A is invertible;
(ii) the nullity of A is 0, i.e., $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
(iii) the rank of A is n;
(iv) for some n-dimensional column vector $\mathbf{b}$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution (which is $\mathbf{x} = A^{-1}\mathbf{b}$);
(v) the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n-dimensional column vector $\mathbf{b}$;
(vi) the row echelon form of A has no zero rows;
(vii) the reduced row echelon form of A is the identity matrix;
(viii) A is a product of elementary matrices.

Properties of reduced row echelon form

Theorem 1 For any matrix, the reduced row echelon form exists and is unique.

Theorem 2 Suppose A and B are matrices of the same dimensions. Then the following conditions are equivalent:

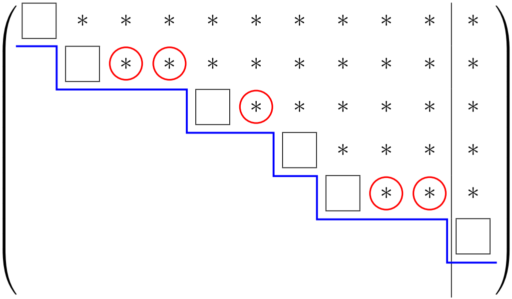
(i) A and B share a reduced row echelon form;
(ii) A and B share a row echelon form;
(iii) A can be obtained from B by applying elementary row operations;
(iv) $A = CB$ for an invertible matrix C;
(v) A and B have the same row space;
(vi) A and B have the same null-space.

How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to **row echelon form**.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.

Consistency check

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the row echelon form of the augmented matrix. This is equivalent to the augmented matrix having the same rank as the coefficient matrix.



Augmented matrix of an inconsistent system

Example. $\begin{cases} x_2 + 2x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \end{cases}$

Variables: x_1, x_2, x_3, x_4.

Augmented matrix: $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 10 \end{array}\right)$

To get it into row echelon form, we exchange the two rows:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array}\right)$$

Consistency check is passed. To convert into reduced row echelon form, add -2 times the 2nd row to the 1st row:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array}\right)$$

The leading variables are x_1 and x_2; hence x_3 and x_4 are free variables.

Back to the system:

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

In vector form, $(x_1, x_2, x_3, x_4) =$

$$= (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).$$

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A \neq 0$ if and only if a system of linear equations with the coefficient matrix A has a unique solution. Equivalently, $\det A \neq 0$ if and only if the matrix A is invertible.

Definition in low dimensions

Definition. $\det(a) = a$, $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+: \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-: \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$