MATH 423

Linear Algebra II

Lecture 18:
Determinants (continued).

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A \neq 0$ if and only if a system of linear equations with the coefficient matrix A has a unique solution. Equivalently, $\det A \neq 0$ if and only if the matrix A is invertible.

Definition in low dimensions

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+: \quad \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-: \quad \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Examples: 3×3 matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3$$

$$- 0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5,$$

$$\begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0$$

$$- 6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6.$$

Let us try to find a solution of a general system of 2 linear equations in 2 variables:

$$\begin{cases} a_{11}x + a_{12}y = b_1, \\ a_{21}x + a_{22}y = b_2. \end{cases}$$

Solve the 1st equation for x: $x = (b_1 - a_{12}y)/a_{11}$.
Substitute into the 2nd equation:

$$a_{21}(b_1 - a_{12}y)/a_{11} + a_{22}y = b_2.$$

Solve for y: $y = \dfrac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$.

Back substitution: $x = (b_1 - a_{12}y)/a_{11} = \dfrac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$.

Thus

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

General definition

The general definition of the determinant is quite complicated as there is no simple explicit formula.

There are several approaches to defining determinants.

Approach 1 (original): an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n-1)\times(n-1)$ matrices.

Original definition of determinant

Definition. If $A = (a_{ij})$ is an $n \times n$ matrix then

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, a_{1,\pi(1)}\, a_{2,\pi(2)} \dots a_{n,\pi(n)},$$

where π runs over S_n, the set of all permutations of $\{1, 2, \dots, n\}$, and $\operatorname{sgn}(\pi)$ denotes the sign of the permutation π.

Remarks. • A **permutation** of the set $\{1, 2, \dots, n\}$ is an invertible mapping of this set onto itself. There are $n!$ such mappings.

• The **sign** $\operatorname{sgn}(\pi)$ can be 1 or -1. Its definition is rather complicated.

$\mathcal{M}_{n,n}(\mathbb{F})$: the set of $n\times n$ matrices with entries in $\mathbb{F}$.

Theorem There exists a unique function $\det : \mathcal{M}_{n,n}(\mathbb{F}) \to \mathbb{F}$ (called the determinant) with the following properties:

(D1) if we interchange two rows of a matrix, the determinant changes its sign;

(D2) if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;

(D3) if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

(D4) $\det I = 1$.

Corollary 1 Suppose A is a square matrix and B is obtained from A applying elementary row operations. Then $\det A = 0$ if and only if $\det B = 0$.

Corollary 2 $\det B = 0$ whenever the matrix B has a zero row.

Hint: Multiply the zero row by the zero scalar.

Corollary 3 $\det A = 0$ if and only if the matrix A is not invertible.

Idea of the proof: Let B be the reduced row echelon form of A. If A is invertible then $B = I$; otherwise B has a zero row.

Remark. The same argument proves that properties (D1)–(D4) are enough to compute any determinant.

Row echelon form of a square matrix A:

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

$$\det A \neq 0$$

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

$$\det A = 0$$

Other properties of determinants

- If a matrix A has two identical rows then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

- If a matrix A has two proportional rows then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ra_1 & ra_2 & ra_3 \end{vmatrix} = r\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

- If the rows of A are linearly dependent then $\det A = 0$ (as in this case A is not invertible).

Definition. A square matrix $A = (a_{ij})$ is called **diagonal** if all entries off the main diagonal are zeros: $a_{ij} = 0$ whenever $i \neq j$. The matrix A is called **upper triangular** if all entries below the main diagonal are zeros: $a_{ij} = 0$ whenever $i > j$.

- If A is a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$ then $\det A = d_1 d_2 \ldots d_n$.

- The determinant of an upper triangular matrix is equal to the product of its diagonal entries.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

Additive law for rows

- Suppose that matrices X, Y, Z are identical except for the ith row and the ith row of Z is the sum of the ith rows of X and Y.

  Then $\boxed{\det Z = \det X + \det Y.}$

$$\begin{vmatrix} a_1+a_1' & a_2+a_2' & a_3+a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Together with property (D2), this means that the determinant depends linearly on each row of a matrix.

Submatrices

Definition. Given a matrix A, a $k \times k$ **submatrix** of A is a matrix obtained by specifying k columns and k rows of A and deleting the other columns and rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \\ 3 & 5 & 7 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} * & 2 & * & 4 \\ * & * & * & * \\ * & 5 & * & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 4 \\ 5 & 9 \end{pmatrix}$$

Theorem Suppose A is a matrix of rank m. Then A admits a $k \times k$ submatrix with nonzero determinant if and only if $0 < k \leq m$.

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the ith row and the jth column of A.

Theorem For any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},$$

(expansion by kth row)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.$$

(expansion by mth column)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Expansion by the 1st row:

$$\begin{pmatrix} \boxed{1} & * & * \\ * & 5 & 6 \\ * & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} * & * & \boxed{3} \\ 4 & 5 & * \\ 7 & 8 & * \end{pmatrix}$$

$$\det A = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Expansion by the 2nd column:

$$\begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ * & \boxed{5} & * \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ 4 & * & 6 \\ * & \boxed{8} & * \end{pmatrix}$$

$$\det A = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$= -2(4 \cdot 9 - 6 \cdot 7) + 5(1 \cdot 9 - 3 \cdot 7) - 8(1 \cdot 6 - 3 \cdot 4) = 0.$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

Subtract the 1st row from the 2nd row and from the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

since the last matrix has two proportional rows.