MATH 423

Linear Algebra II

Lecture 19: More on determinants.

Determinants: definition in low dimensions

Definition. $\det(a) = a$, $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+: \quad \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-: \quad \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Properties of determinants

Determinants and elementary row operations:

- if we interchange two rows of a matrix, the determinant changes its sign;
- if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same.

Properties of determinants

Tests for non-invertibility:

- if a matrix A has a zero row then $\det A = 0$;
- if a matrix A has two identical rows then $\det A = 0$;
- if a matrix A has two proportional rows then $\det A = 0$;
- if the rows of a matrix A are linearly dependent vectors then $\det A = 0$.

Properties of determinants

Special matrices:

- $\det I = 1$;
- the determinant of a diagonal matrix is equal to the product of its diagonal entries;
- the determinant of an upper triangular matrix is equal to the product of its diagonal entries.

Characterization of determinants

Theorem 1 The determinant is the only function on $\mathcal{M}_{n,n}(\mathbb{F})$ with the following properties:

- it changes the sign when we interchange two rows of a matrix;
- it is multiplied by a scalar r when a row of a matrix is multiplied by r;
- it is not changed when we add a row of a matrix multiplied by a scalar to another row;
- it takes the value 1 at the identity matrix.

Theorem 2 The determinant is the only function on $\mathcal{M}_{n,n}(\mathbb{F})$ with the following properties:

- it depends linearly on each row of a matrix;
- it takes the value 0 at any matrix with two identical rows;
- it takes the value 1 at the identity matrix.

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column of A.

Theorem For any $1 \le k, m \le n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},$$

(*expansion by* k*th row*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.$$

(*expansion by* m*th column*)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Evaluation of determinants

Example. $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{pmatrix}.$

First let's do some row reduction.

Add -4 times the 1st row to the 2nd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix}$$

Add -7 times the 1st row to the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

Expand the determinant by the 1st column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix}$$

Thus

$$\det B = \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix}$$

$$= (-3)(-2) \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-3)(-2)(-2) = -12.$$

Example. $C = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}$, $\det C = ?$

Expand the determinant by the 3rd column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Add -2 times the 2nd row to the 1st row:

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det C = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \cdot 9 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

Thus

$$\det C = -18 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18 \cdot 2 = -36.$$

More properties of determinants

Determinants and matrix multiplication:

- if A and B are $n \times n$ matrices then
$$\det(AB) = \det A \cdot \det B;$$
- if A and B are $n \times n$ matrices then
$$\det(AB) = \det(BA);$$
- if A is an invertible matrix then
$$\det(A^{-1}) = (\det A)^{-1}.$$

Determinants and scalar multiplication:

- if A is an $n \times n$ matrix and $r \in \mathbb{F}$ then
$$\det(rA) = r^n \det A.$$

Theorem $\det(AB) = \det A \cdot \det B$ for any $n \times n$ matrices A and B.

Proof: First we consider a special case when $A = E_\sigma$ is an elementary matrix corresponding to an elementary row operation σ. It is easy to observe that $\det(E_\sigma B) = r_\sigma \det B$ for some scalar r_σ depending only on σ. For $B = I$, we get $\det E_\sigma = r_\sigma \det I = r_\sigma$. Hence $\det(E_\sigma B) = (\det E_\sigma)(\det B)$.

Now it follows by induction that

$$\det(E_k E_{k-1} \dots E_2 E_1 B) = (\det E_k) \dots (\det E_2)(\det E_1)(\det B)$$

for any elementary matrices $E_1, E_2, \dots, E_k$ and any B.
Again, for $B = I$ we get

$$\det(E_k E_{k-1} \dots E_2 E_1) = (\det E_k) \dots (\det E_2)(\det E_1).$$

Therefore $\det(AB) = (\det A)(\det B)$ whenever A is a product of elementary matrices. That is, whenever A is invertible.

It remains to consider the case when A is not invertible. In this case, $\mathrm{rank}(A) < n$. Then $\mathrm{rank}(AB) \leq \mathrm{rank}(A) < n$ so that AB is not invertible too. Thus $\det(AB) = \det A = 0$.

Determinant of the transpose

Theorem $\det A^t = \det A$ for any square matrix A.

Proof: First we consider a special case when $A = E_\sigma$, an elementary matrix associated to an elementary row operation σ. If σ exchanges two rows or multiplies a row by a scalar then $E_\sigma^t = E_\sigma$. If σ adds a scalar multiple of one row to another row, then E_σ^t is also an elementary matrix associated to an operation of the same type. Hence $\det E_\sigma^t = \det E_\sigma = 1$.

Now consider the case when A is invertible. In this case A is a product of elementary matrices, $A = E_k \dots E_2 E_1$. Then $A^t = E_1^t E_2^t \dots E_k^t$. We have $\det A = (\det E_k) \dots (\det E_2)(\det E_1)$, $\det A^t = (\det E_1^t)(\det E_2^t) \dots (\det E_k^t)$. Since $\det E_i^t = \det E_i$ for all i, we obtain $\det A^t = \det A$.

It remains to consider the case when A is not invertible. Since $\operatorname{rank}(A^t) = \operatorname{rank}(A)$, the transpose A^t is not invertible too. Thus in this case $\det A^t = \det A = 0$.

Columns vs. rows

Since $\det A^t = \det A$, for every property of determinants involving rows of a matrix there is an analogous property involving columns of a matrix.

- If one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.
- Interchanging two columns of a matrix changes the sign of its determinant.
- If a matrix A has two proportional columns then $\det A = 0$.
- Adding a scalar multiple of one column to another does not change the determinant of a matrix.

Determinants and the inverse matrix

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column of A. The **cofactor matrix** of A is an $n \times n$ matrix $\widetilde{A} = (\alpha_{ij})$ defined by $\alpha_{ij} = (-1)^{i+j} \det M_{ij}$.

Theorem $\widetilde{A}^t A = A \widetilde{A}^t = (\det A) I$.

Sketch of the proof: $A \widetilde{A}^t = (\det A) I$ means that

$$\sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj} = \det A \quad \text{for all } k,$$

$$\sum_{j=1}^{n} (-1)^{k+j} a_{mj} \det M_{kj} = 0 \quad \text{for } m \neq k.$$

Indeed, the 1st equality is the expansion of $\det A$ by the kth row. The 2nd equality is an analogous expansion of $\det B$, where the matrix B is obtained from A by replacing its kth row with a copy of the mth row (clearly, $\det B = 0$). $\widetilde{A}^t A = (\det A) I$ is verified similarly, using column expansions.

Corollary If $\det A \neq 0$ then $A^{-1} = (\det A)^{-1} \widetilde{A}^t$.