MATH 423

Linear Algebra II

Lecture 20:
Geometry of linear transformations. Eigenvalues and eigenvectors. Characteristic polynomial.

Geometric properties of determinants

- 2×2 determinants and plane geometry

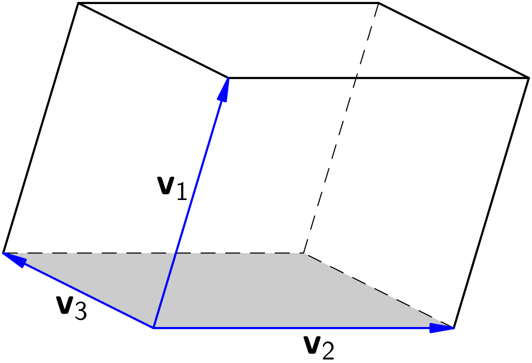
Let P be a parallelogram in the plane $\mathbb{R}^2$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are represented by adjacent sides of P. Then $\operatorname{area}(P) = |\det A|$, where $A = (\mathbf{v}_1, \mathbf{v}_2)$, a matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$.

Consider a linear operator $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_A(\mathbf{v}) = A\mathbf{v}$ for any column vector $\mathbf{v}$. Then $\operatorname{area}(L_A(D)) = |\det A| \operatorname{area}(D)$ for any bounded domain D.

- 3×3 determinants and space geometry

Let Π be a parallelepiped in space $\mathbb{R}^3$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are represented by adjacent edges of Π. Then $\operatorname{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, a matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Similarly, $\operatorname{volume}(L_B(D)) = |\det B| \operatorname{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$.



$\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Note that the parallelepiped Π is the image under L_B of a unit cube whose adjacent edges are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

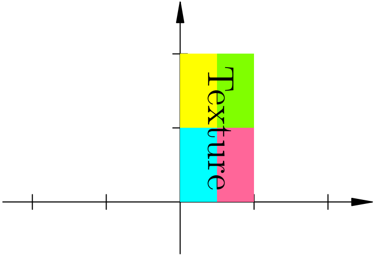
The triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the right-hand rule. We say that L_B **preserves orientation** if it preserves the hand rule for any basis. This is the case if and only if $\det B > 0$.

Linear transformations of $\mathbb{R}^2$

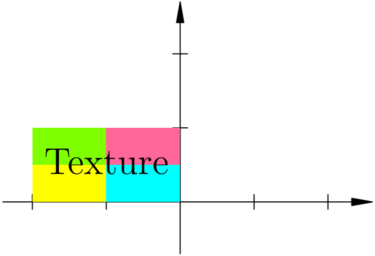
Any linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ is represented as multiplication of a 2-dimensional column vector by a 2×2 matrix: $L(\mathbf{x}) = A\mathbf{x}$ or

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

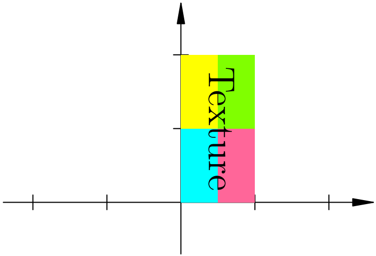
Linear transformations corresponding to particular matrices can have various geometric properties.



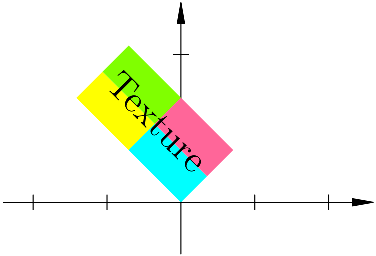
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



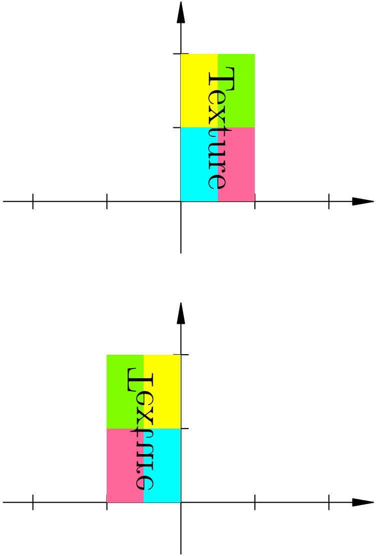
Rotation by 90^{o}



$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

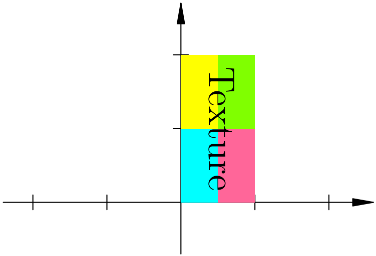


Rotation by 45^{o}

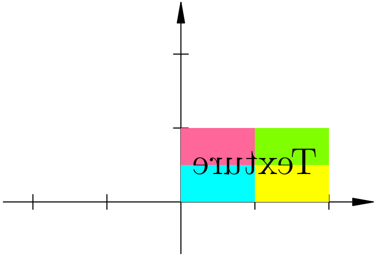


$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

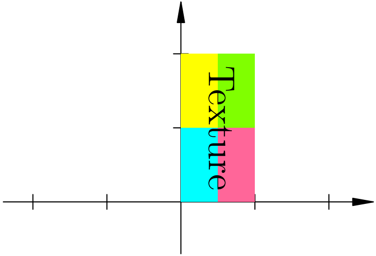
Reflection about
the vertical axis



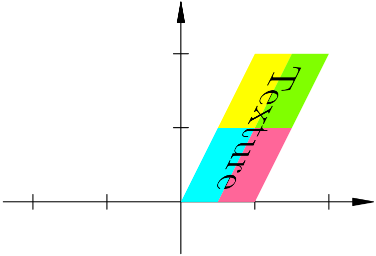
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



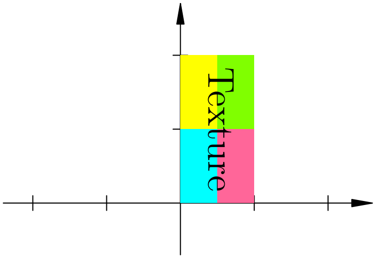
Reflection about the line $x - y = 0$



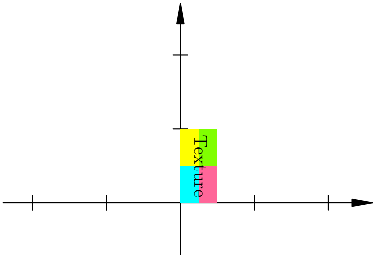
$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$



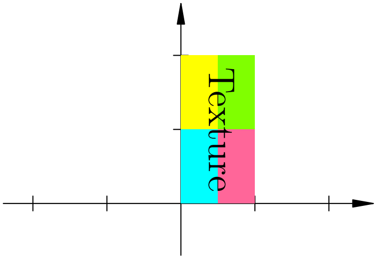
Horizontal shear



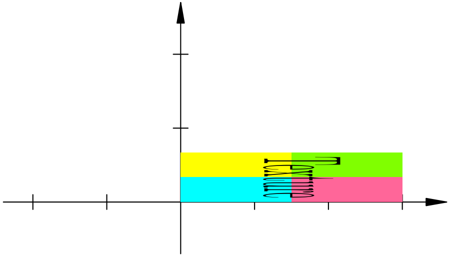
$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$



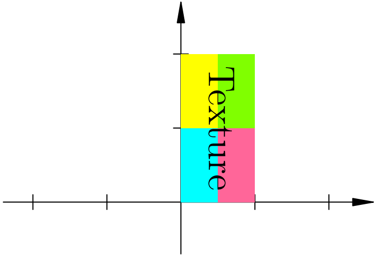
Scaling



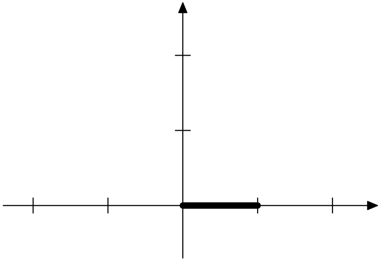
$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$



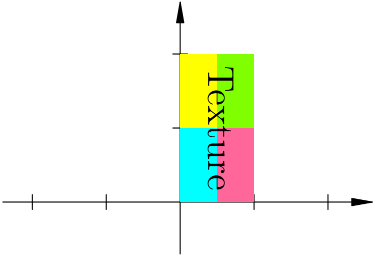
Squeeze



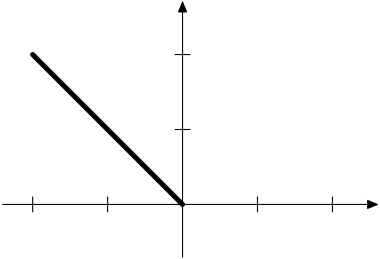
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



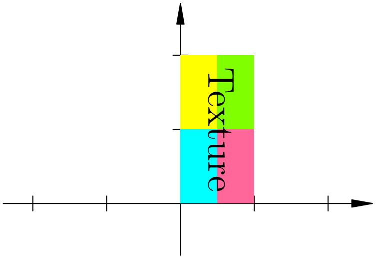
Vertical projection on
the horizontal axis



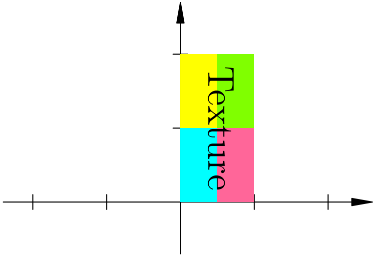
$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$



Horizontal projection
on the line $x + y = 0$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Identity

Eigenvalues and eigenvectors

Definition. Let $A \in \mathcal{M}_{n,n}(\mathbb{F})$. A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of the matrix A if $\boxed{A\mathbf{v} = \lambda\mathbf{v}}$ for a nonzero column vector $\mathbf{v} \in \mathbb{F}^n$.

The vector $\mathbf{v}$ is called an **eigenvector** of A belonging to (or associated with) the eigenvalue λ.

Remarks. • Alternative notation:
eigenvalue = **characteristic value**,
eigenvector = **characteristic vector**.

• The zero vector is never considered an eigenvector.

Example. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A belonging to the eigenvalue 1, while $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A belonging to the eigenvalue -1.

Eigenspaces

Let A be an $n \times n$ matrix. Let $\mathbf{v}$ be an eigenvector of A belonging to an eigenvalue λ. Then

$$A\mathbf{v} = \lambda\mathbf{v} \implies A\mathbf{v} = (\lambda I)\mathbf{v} \implies (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Hence $\mathbf{v} \in \mathcal{N}(A - \lambda I)$, the null-space of the matrix $A - \lambda I$.

Conversely, if $\mathbf{x} \in \mathcal{N}(A - \lambda I)$ then $A\mathbf{x} = \lambda\mathbf{x}$. Thus the eigenvectors of A belonging to the eigenvalue λ are nonzero vectors from $\mathcal{N}(A - \lambda I)$.

Definition. If $\mathcal{N}(A - \lambda I) \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of the matrix A corresponding to the eigenvalue λ.

How to find eigenvalues and eigenvectors?

Theorem Given a square matrix A and a scalar λ, the following conditions are equivalent:

- λ is an eigenvalue of A,
- $\mathcal{N}(A - \lambda I) \neq \{\mathbf{0}\}$,
- the matrix $A - \lambda I$ is not invertible,
- $\det(A - \lambda I) = 0$.

Definition. $\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A.

Eigenvalues λ of A are roots of the characteristic equation. Associated eigenvectors of A are nonzero solutions of the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Example. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc). \end{aligned}$$

Example. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

$$= -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3,$$

where $c_1 = a_{11} + a_{22} + a_{33}$ (the *trace* of A),

$$c_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$c_3 = \det A.$

Theorem. Let $A = (a_{ij})$ be an $n \times n$ matrix. Then $\det(A - \lambda I)$ is a polynomial of λ of degree n:

$$\det(A - \lambda I) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n.$$

Furthermore, $(-1)^{n-1} c_1 = a_{11} + a_{22} + \cdots + a_{nn}$ and $c_n = \det A$.

Definition. The polynomial $p(\lambda) = \det(A - \lambda I)$ is called the **characteristic polynomial** of the matrix A.

Corollary Any $n \times n$ matrix has at most n eigenvalues.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Characteristic equation: $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0.$

$(2-\lambda)^2 - 1 = 0 \implies \lambda_1 = 1, \ \lambda_2 = 3.$

$$(A - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + y = 0.$$

The general solution is $(-t, t) = t(-1, 1), \ t \in \mathbb{R}$. Thus $\mathbf{v}_1 = (-1, 1)$ is an eigenvector associated with the eigenvalue 1. The corresponding eigenspace is the line spanned by $\mathbf{v}_1$.

$$(A - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x - y = 0.$$

The general solution is $(t, t) = t(1, 1), \ t \in \mathbb{R}$.

Thus $\mathbf{v}_2 = (1, 1)$ is an eigenvector associated with the eigenvalue 3. The corresponding eigenspace is the line spanned by $\mathbf{v}_2$.

Summary. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$

- The matrix A has two eigenvalues: 1 and 3.
- The eigenspace of A associated with the eigenvalue 1 is the line $t(-1, 1)$.
- The eigenspace of A associated with the eigenvalue 3 is the line $t(1, 1)$.
- Eigenvectors $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 1)$ of the matrix A are orthogonal and form a basis for $\mathbb{R}^2$.
- Geometrically, the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is a stretch by a factor of 3 away from the line $x + y = 0$ in the orthogonal direction.

Eigenvalues and eigenvectors of an operator

Definition. Let V be a vector space and $L : V \to V$ be a linear operator. A scalar λ is called an **eigenvalue** of the operator L if $\boxed{L(\mathbf{v}) = \lambda\mathbf{v}}$ for a nonzero vector $\mathbf{v} \in V$. The vector $\mathbf{v}$ is called an **eigenvector** of L associated with the eigenvalue λ.

If $V = \mathbb{F}^n$ then the linear operator L is given by $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n\times n$ matrix. In this case, eigenvalues and eigenvectors of the operator L are precisely eigenvalues and eigenvectors of the matrix A.

For a general finite-dimensional vector space V, we choose an ordered basis α. Then

$$L(\mathbf{v}) = \lambda\mathbf{v} \iff [L]_\alpha[\mathbf{v}]_\alpha = \lambda[\mathbf{v}]_\alpha.$$

Hence the eigenvalues of L coincide with those of the matrix $[L]_\alpha$. Moreover, the associated eigenvectors of $[L]_\alpha$ are coordinates of the eigenvectors of L.