MATH 423

Linear Algebra II

Lecture 26:
Review for Test 2.

Topics for Test 2

Elementary row operations (F/I/S 3.1–3.4)

- Elementary row operations
- Reduced row echelon form
- Solving systems of linear equations
- Computing the inverse matrix

Determinants (F/I/S 4.1–4.5)

- Definition for 2×2 and 3×3 matrices
- Properties of determinants
- Row and column expansions
- Evaluation of determinants

Topics for Test 2

Eigenvalues and eigenvectors (F/I/S 5.1–5.4)

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Diagonalization, basis of eigenvectors
- Matrix polynomials
- Markov chains, limit distributions
- Cayley-Hamilton Theorem

Sample problems for Test 2

Problem 1 (20 pts.) Find a cubic polynomial $p(x)$ such that $p(-2) = 0$, $p(-1) = 4$, $p(1) = 0$, and $p(2) = 4$.

Problem 2 (25 pts.) Evaluate a determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 \end{vmatrix}.$$

For which values of parameters c_1, c_2, c_3, c_4 is this determinant equal to zero?

Sample problems for Test 2

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.
(ii) For each eigenvalue of A, find an associated eigenvector.
(iii) Find all eigenvalues of the matrix A^3.

Problem 4 (25 pts.) Let $B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. Find a matrix C such that $C^2 = B^2$, but $C \neq \pm B$.

Sample problems for Test 2

Bonus Problem 5 (15 pts.) Let X be a square matrix that can be represented as a block matrix

$$X = \begin{pmatrix} A & C \\ O & B \end{pmatrix},$$

where A and B are square matrices and O is a zero matrix. Prove that $\det(X) = \det(A)\det(B)$.

Problem 1. Find a cubic polynomial $p(x)$ such that $p(-2) = 0$, $p(-1) = 4$, $p(1) = 0$, and $p(2) = 4$.

Let $p(x) = a + bx + cx^2 + dx^3$. Then

$$\begin{aligned} p(-2) &= a - 2b + 4c - 8d, \\ p(-1) &= a - b + c - d, \\ p(1) &= a + b + c + d, \\ p(2) &= a + 2b + 4c + 8d. \end{aligned}$$

The coefficients a, b, c, and d are to be chosen so that

$$\begin{cases} a - 2b + 4c - 8d = 0, \\ a - b + c - d = 4, \\ a + b + c + d = 0, \\ a + 2b + 4c + 8d = 4. \end{cases}$$

This is a system of linear equations. To solve it, we convert the augmented matrix to reduced row echelon form using elementary row operations.

$$
\left(\begin{array}{rrrr|r}
1 & -2 & 4 & -8 & 0 \\
1 & -1 & 1 & -1 & 4 \\
1 & 1 & 1 & 1 & 0 \\
1 & 2 & 4 & 8 & 4
\end{array}\right)
\longrightarrow
\left(\begin{array}{rrrr|r}
1 & 1 & 1 & 1 & 0 \\
1 & -1 & 1 & -1 & 4 \\
1 & -2 & 4 & -8 & 0 \\
1 & 2 & 4 & 8 & 4
\end{array}\right)
$$

$$
\longrightarrow
\left(\begin{array}{rrrr|r}
1 & 1 & 1 & 1 & 0 \\
0 & -2 & 0 & -2 & 4 \\
1 & -2 & 4 & -8 & 0 \\
1 & 2 & 4 & 8 & 4
\end{array}\right)
\longrightarrow
\left(\begin{array}{rrrr|r}
1 & 1 & 1 & 1 & 0 \\
0 & -2 & 0 & -2 & 4 \\
0 & -3 & 3 & -9 & 0 \\
0 & 1 & 3 & 7 & 4
\end{array}\right)
$$

$$
\longrightarrow
\left(\begin{array}{rrrr|r}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & -2 \\
0 & 1 & -1 & 3 & 0 \\
0 & 1 & 3 & 7 & 4
\end{array}\right)
\longrightarrow
\left(\begin{array}{rrrr|r}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & -2 \\
0 & 0 & -1 & 2 & 2 \\
0 & 0 & 3 & 6 & 6
\end{array}\right)
$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 2 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

It follows that $a = 2$, $b = -3$, $c = 0$, and $d = 1$.
Thus $p(x) = x^3 - 3x + 2$.

Problem 2. Evaluate a determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 \end{vmatrix}$.

For which values of parameters c_1, c_2, c_3, c_4 is this determinant equal to zero?

Let d denote the value of the determinant. To find d, we use a nonstandard row reduction. We subtract c_1 times the 3rd row from the 4th row, then subtract c_1 times the 2nd row from the 3rd row, then subtract c_1 times the 1st row from the 2nd row:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & c_2 - c_1 & c_3 - c_1 & c_4 - c_1 \\ 0 & c_2^2 - c_1 c_2 & c_3^2 - c_1 c_3 & c_4^2 - c_1 c_4 \\ 0 & c_2^3 - c_1 c_2^2 & c_3^3 - c_1 c_3^2 & c_4^3 - c_1 c_4^2 \end{vmatrix}.$$

The expansion by the first column yields

$$d = \begin{vmatrix} c_2 - c_1 & c_3 - c_1 & c_4 - c_1 \\ c_2^2 - c_1 c_2 & c_3^2 - c_1 c_3 & c_4^2 - c_1 c_4 \\ c_2^3 - c_1 c_2^2 & c_3^3 - c_1 c_3^2 & c_4^3 - c_1 c_4^2 \end{vmatrix}.$$

Now there is a common factor in each column:

$$d = \begin{vmatrix} c_2 - c_1 & c_3 - c_1 & c_4 - c_1 \\ (c_2 - c_1)c_2 & (c_3 - c_1)c_3 & (c_4 - c_1)c_4 \\ (c_2 - c_1)c_2^2 & (c_3 - c_1)c_3^2 & (c_4 - c_1)c_4^2 \end{vmatrix}$$

$$= (c_2 - c_1)(c_3 - c_1)(c_4 - c_1) \begin{vmatrix} 1 & 1 & 1 \\ c_2 & c_3 & c_4 \\ c_2^2 & c_3^2 & c_4^2 \end{vmatrix}.$$

The latter determinant is evaluated using the same technique as before. Eventually we get

$$d = (c_2 - c_1)(c_3 - c_1)(c_4 - c_1)(c_3 - c_2)(c_4 - c_2)(c_4 - c_3).$$

The determinant is equal to zero if and only if the numbers c_1, c_2, c_3, c_4 are not all distinct.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.

The eigenvalues of A are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 - 2(1-\lambda) - 2(1-\lambda) = (1-\lambda)\big((1-\lambda)^2 - 4\big)$$

$$= (1-\lambda)\big((1-\lambda) - 2\big)\big((1-\lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix A has three eigenvalues: -1, 1, and 3.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(ii) For each eigenvalue of A, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)$ of the matrix A associated with an eigenvalue λ is a nonzero solution of the vector equation

$$(A-\lambda I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve the equation, we convert the matrix $A - \lambda I$ to reduced row echelon form.

First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)$ is an eigenvector of A associated with the eigenvalue -1.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)$ is an eigenvector of A associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$A - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - 3I)\mathbf{v} = \mathbf{0} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of A associated with the eigenvalue 3.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iii) Find all eigenvalues of the matrix A^3.

Suppose that $\mathbf{v}$ is an eigenvector of the matrix A associated with an eigenvalue λ, that is, $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v},$$
$$A^3\mathbf{v} = A(A^2\mathbf{v}) = A(\lambda^2\mathbf{v}) = \lambda^2(A\mathbf{v}) = \lambda^2(\lambda\mathbf{v}) = \lambda^3\mathbf{v}.$$

Therefore $\mathbf{v}$ is also an eigenvector of the matrix A^3 and the associated eigenvalue is λ^3. We already know that the matrix A has eigenvalues -1, 1, and 3. It follows that A^3 has eigenvalues -1, 1, and 27.

It remains to notice that a 3×3 matrix can have at most 3 eigenvalues.

Problem 4. Let $B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. Find a matrix C such that $C^2 = B^2$, but $C \neq \pm B$.

This problem is simple in the case B is diagonal. Indeed, if $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \neq 0$, then we can take

$$C = \begin{pmatrix} -a & 0 \\ 0 & b \end{pmatrix} \quad \text{or} \quad C = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}.$$

Therefore the diagonalization of the matrix B might help. The characteristic polynomial of B is

$$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3$$

$$= \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5).$$

The eigenvalues are 1 and 5.

An eigenvector for the eigenvalue 1 is $\mathbf{v}_1 = (-3, 1)$.
An eigenvector for the eigenvalue 5 is $\mathbf{v}_2 = (1, 1)$.
The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$. It follows that $B = UDU^{-1}$, where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad U = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now we let $C = UPU^{-1}$, where $P = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$.

By construction, $P^2 = D^2$ and $P \neq \pm D$. Since $C^2 = UPU^{-1}UPU^{-1} = UP^2U^{-1}$ and, similarly, $B^2 = UD^2U^{-1}$, we obtain that $C^2 = B^2$ and $C \neq \pm B$.

It remains to compute the matrix C:

$$C = UPU^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 9 \\ 3 & 7 \end{pmatrix}.$$

Bonus Problem 5. Let X be a square matrix that can be represented as a block matrix $X = \begin{pmatrix} A & C \\ O & B \end{pmatrix}$, where A and B are square matrices and O is a zero matrix. Prove that $\det(X) = \det(A)\det(B)$.

Consider block matrices $Y = \begin{pmatrix} I & C \\ O & B \end{pmatrix}$, $Z = \begin{pmatrix} A & O' \\ O & I' \end{pmatrix}$,

where I and I' are identity matrices and O' is a zero matrix. Multiplying Y and Z as block matrices, we obtain

$$YZ = \begin{pmatrix} IA + CO & IO' + CI' \\ OA + BO & OO' + BI' \end{pmatrix} = \begin{pmatrix} A & C \\ O & B \end{pmatrix} = X.$$

As a consequence, $\det(X) = \det(Y)\det(Z)$.

It remains to show that $\det(Y) = \det(B)$ and $\det(Z) = \det(A)$. The first equality is established by repeatedly expanding the determinant of Y along the first column. To get the second equality, we expand the determinant of Z along the last row.