MATH 423

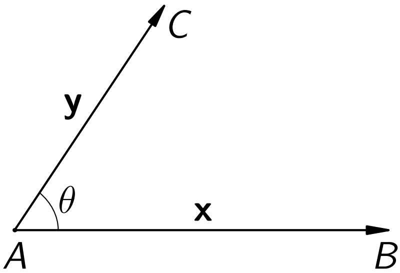
Linear Algebra II

Lecture 27:
Norms and inner products.

Euclidean structure

In addition to the linear structure (addition and scaling), space $\mathbb{R}^3$ carries the Euclidean structure:

- length of a vector: $|\mathbf{x}|$,
- angle between vectors: θ,
- dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\cos\theta$.



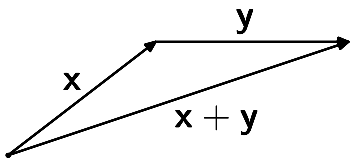
Euclidean structure

Properties of vector length:

$|\mathbf{x}| \geq 0, \;\; |\mathbf{x}| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$|r\mathbf{x}| = |r|\,|\mathbf{x}|$ (homogeneity)

$|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ (triangle inequality)



Euclidean structure

Properties of dot product:

$$\mathbf{x}\cdot\mathbf{x}\geq 0,\ \ \mathbf{x}\cdot\mathbf{x}=0 \text{ only if } \mathbf{x}=\mathbf{0} \qquad \text{(positivity)}$$

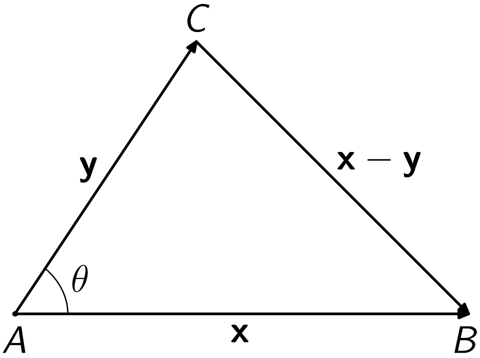
$$\mathbf{x}\cdot\mathbf{y}=\mathbf{y}\cdot\mathbf{x} \qquad \text{(symmetry)}$$

$$(\mathbf{x}+\mathbf{y})\cdot\mathbf{z}=\mathbf{x}\cdot\mathbf{z}+\mathbf{y}\cdot\mathbf{z} \qquad \text{(distributive law)}$$

$$(r\mathbf{x})\cdot\mathbf{y}=r(\mathbf{x}\cdot\mathbf{y}) \qquad \text{(homogeneity)}$$

Relations between lengths and dot products:

- $|\mathbf{x}|=\sqrt{\mathbf{x}\cdot\mathbf{x}}$
- $\mathbf{x}\cdot\mathbf{y}=\frac{1}{2}\big(|\mathbf{x}|^2+|\mathbf{y}|^2-|\mathbf{x}-\mathbf{y}|^2\big)$



Law of cosines:

$$|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}|\,|\mathbf{y}|\,\cos\theta$$

$$|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\,\mathbf{x}\cdot\mathbf{y}$$

Norm

The notion of *norm* generalizes the notion of length of a vector in $\mathbb{R}^n$.

Definition. Let V be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A function $\alpha : V \to \mathbb{R}$ is called a **norm** on V if it has the following properties:

(i) $\alpha(\mathbf{x}) \geq 0$, $\alpha(\mathbf{x}) = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\alpha(r\mathbf{x}) = |r|\,\alpha(\mathbf{x})$ for all $r \in \mathbb{F}$ (homogeneity)
(iii) $\alpha(\mathbf{x}+\mathbf{y}) \leq \alpha(\mathbf{x}) + \alpha(\mathbf{y})$ (triangle inequality)

Notation. The norm of a vector $\mathbf{x} \in V$ is usually denoted $\|\mathbf{x}\|$. Different norms on V are distinguished by subscripts, e.g., $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$.

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$.

Positivity and homogeneity are obvious. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$.

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \max_j |x_j| + \max_j |y_j|$$
$$\implies \max_j |x_j + y_j| \leq \max_j |x_j| + \max_j |y_j|$$
$$\implies \|\mathbf{x} + \mathbf{y}\|_\infty \leq \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.$$

- $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.

Positivity and homogeneity are obvious.
The triangle inequality: $|x_i + y_i| \leq |x_i| + |y_i|$

$$\implies \sum_j |x_j + y_j| \leq \sum_j |x_j| + \sum_j |y_j|$$

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_p = \big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\big)^{1/p}$, $p > 0$.

Remark. $\|\mathbf{x}\|_2$ = Euclidean length of $\mathbf{x}$.

Theorem $\|\mathbf{x}\|_p$ is a norm on $\mathbb{R}^n$ for any $p \geq 1$.

Positivity and homogeneity are still obvious (and hold for any $p > 0$). The triangle inequality for $p \geq 1$ is known as the **Minkowski inequality**:

$$\big(|x_1 + y_1|^p + |x_2 + y_2|^p + \cdots + |x_n + y_n|^p\big)^{1/p} \leq$$
$$\leq \big(|x_1|^p + \cdots + |x_n|^p\big)^{1/p} + \big(|y_1|^p + \cdots + |y_n|^p\big)^{1/p}.$$

Normed vector space

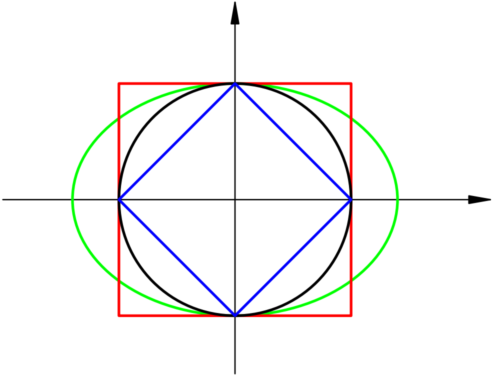
Definition. A **normed vector space** is a vector space endowed with a norm.

The norm defines a distance function on the normed vector space: $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

Then we say that a sequence $\mathbf{x}_1, \mathbf{x}_2, \dots$ *converges* to a vector $\mathbf{x}$ if $\mathrm{dist}(\mathbf{x}, \mathbf{x}_n) \to 0$ as $n \to \infty$.

Also, we say that a vector $\mathbf{x}$ is a good *approximation* of a vector $\mathbf{x}_0$ if $\mathrm{dist}(\mathbf{x}, \mathbf{x}_0)$ is small.

Unit circle: $\|\mathbf{x}\| = 1$



$\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$	black
$\|\mathbf{x}\| = \left(\frac{1}{2}x_1^2 + x_2^2\right)^{1/2}$	green
$\|\mathbf{x}\| = \|x_1\| + \|x_2\|$	blue
$\|\mathbf{x}\| = \max(\|x_1\|, \|x_2\|)$	red

Theorem 1 Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Then there exist positive constants c_1, c_2 such that

$$c_1\|\mathbf{x}\|_2 \leq \|\mathbf{x}\| \leq c_2\|\mathbf{x}\|_2 \quad \text{for any} \quad \mathbf{x} \in \mathbb{R}^n.$$

Idea of the proof: One shows that the function $f(\mathbf{x}) = \|\mathbf{x}\|$ is continuous on $\mathbb{R}^n$ (in the usual sense). Since the Euclidean unit sphere S^{n-1} (given by $\|\mathbf{x}\|_2 = 1$) is a closed bounded set, the function f attains its minimum and maximum values on S^{n-1}. These are c_1 and c_2 in the above inequalities.

Theorem 2 Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be arbitrary norms on a finite-dimensional vector space V. Then there exist positive constants c_1, c_2 such that

$$c_1\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq c_2\|\mathbf{x}\|_2 \quad \text{for any} \quad \mathbf{x} \in V.$$

Corollary A sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ of vectors from V converges to a vector $\mathbf{x} \in V$ with respect to the distance induced by the norm $\|\cdot\|_1$ if and only if it converges to $\mathbf{x}$ with respect to the distance induced by $\|\cdot\|_2$.

Examples. $V = C[a,b]$, $f : [a,b] \to \mathbb{R}$.

- $\|f\|_\infty = \max_{a \le x \le b} |f(x)|$.
- $\|f\|_1 = \int_a^b |f(x)|\,dx$.
- $\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}$, $p > 0$.

Theorem $\|f\|_p$ is a norm on $C[a,b]$ for any $p \ge 1$.

Inner product: real vector space

The notion of *inner product* generalizes the notion of dot product of vectors in $\mathbb{R}^3$.

Definition. Let V be a real vector space. A function $\beta : V \times V \to \mathbb{R}$, usually denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if it is positive, symmetric, and bilinear. That is, if

(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry)
(iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)
(iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (distributive law)

An **inner product space** is a vector space endowed with an inner product.

Examples. $V = \mathbb{R}^n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.
- $\langle \mathbf{x}, \mathbf{y} \rangle = d_1 x_1 y_1 + d_2 x_2 y_2 + \cdots + d_n x_n y_n$,

where $d_1, d_2, \ldots, d_n > 0$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = (D\mathbf{x}) \cdot (D\mathbf{y})$,

where D is an invertible $n \times n$ matrix.

Remarks. (a) Invertibility of D is necessary to show that $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \implies \mathbf{x} = \mathbf{0}$.

(b) The second example is a particular case of the third one when $D = \mathrm{diag}(d_1^{1/2}, d_2^{1/2}, \ldots, d_n^{1/2})$.

Example. $V = \mathcal{M}_{m,n}(\mathbb{R})$, space of $m \times n$ matrices.

- $\langle A, B \rangle = \mathrm{trace}\,(AB^t)$.

If $A = (a_{ij})$ and $B = (b_{ij})$, then $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$.

Examples. $V = C[a, b]$.

- $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$.

- $\langle f, g \rangle = \int_a^b f(x) g(x) w(x)\, dx$,

where w is bounded, piecewise continuous, and $w > 0$ everywhere on $[a, b]$.

w is called the **weight** function.