

MATH 423

Linear Algebra II

Lecture 28:
Inner product spaces.

Norm

The notion of *norm* generalizes the notion of length of a vector in \mathbb{R}^3 .

Definition. Let V be a vector space over \mathbb{F} , where $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . A function $\alpha : V \rightarrow \mathbb{R}$ is called a **norm** on V if it has the following properties:

- (i) $\alpha(\mathbf{x}) \geq 0$, $\alpha(\mathbf{x}) = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
- (ii) $\alpha(r\mathbf{x}) = |r| \alpha(\mathbf{x})$ for all $r \in \mathbb{F}$ (homogeneity)
- (iii) $\alpha(\mathbf{x} + \mathbf{y}) \leq \alpha(\mathbf{x}) + \alpha(\mathbf{y})$ (triangle inequality)

Notation. The norm of a vector $\mathbf{x} \in V$ is usually denoted $\|\mathbf{x}\|$. Different norms on V are distinguished by subscripts, e.g., $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$.

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|)$.
- $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{1/p}$, $p \geq 1$.

Examples. $V = C[a, b]$, $f : [a, b] \rightarrow \mathbb{R}$.

- $\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|$.
- $\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$, $p \geq 1$.

Inner product: real vector space

The notion of *inner product* generalizes the notion of dot product of vectors in \mathbb{R}^3 .

Definition. Let V be a real vector space. A function $\beta : V \times V \rightarrow \mathbb{R}$, usually denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if it is positive, symmetric, and bilinear. That is, if

- (i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
- (ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry)
- (iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)
- (iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (distributive law)

An **inner product space** is a vector space endowed with an inner product.

Examples. $V = \mathbb{R}^n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = d_1x_1y_1 + d_2x_2y_2 + \cdots + d_nx_ny_n$,

where $d_1, d_2, \dots, d_n > 0$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = (D\mathbf{x}) \cdot (D\mathbf{y})$,

where D is an invertible $n \times n$ matrix.

Example. $V = \mathcal{M}_{m,n}(\mathbb{R})$, space of $m \times n$ matrices.

- $\langle A, B \rangle = \text{trace}(AB^t)$.

If $A = (a_{ij})$ and $B = (b_{ij})$, then $\langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij}b_{ij}$.

Examples. $V = C[a, b]$.

- $\langle f, g \rangle = \int_a^b f(x)g(x) dx.$

- $\langle f, g \rangle = \int_a^b f(x)g(x)w(x) dx,$

where w is bounded, piecewise continuous, and $w > 0$ everywhere on $[a, b]$.

w is called the **weight** function.

Inner product: complex vector space

Definition. Let V be a complex vector space. A function $\beta : V \times V \rightarrow \mathbb{C}$, usually denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if it is positive, conjugate symmetric, and depends linearly on the first argument. That is, if

(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)

(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ (conjugate symmetry)

(iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)

(iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (distributive law)

Dependence on the second argument:

$$\langle \mathbf{x}, r\mathbf{y} + s\mathbf{z} \rangle = \bar{r}\langle \mathbf{x}, \mathbf{y} \rangle + \bar{s}\langle \mathbf{x}, \mathbf{z} \rangle.$$

Example. $V = \mathbb{C}^n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 \overline{y_1} + x_2 \overline{y_2} + \cdots + x_n \overline{y_n}$.

If $z = r + is$, then $\overline{z} = r - is$, $z\overline{z} = r^2 + s^2 = |z|^2$.

Therefore $\langle \mathbf{x}, \mathbf{x} \rangle = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$.

Also, $\overline{\langle \mathbf{x}, \mathbf{y} \rangle} = \overline{x_1 \overline{y_1} + \cdots + x_n \overline{y_n}} = \overline{x_1} \overline{\overline{y_1}} + \cdots + \overline{x_n} \overline{\overline{y_n}}$
 $= \overline{x_1} y_1 + \cdots + \overline{x_n} y_n = \langle \mathbf{y}, \mathbf{x} \rangle$.

Examples. $V = C([a, b], \mathbb{C})$.

- $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$.

- $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} w(x) dx$,

where the weight function w is bounded, piecewise continuous, and $w > 0$ everywhere on $[a, b]$.

Theorem Suppose $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on a vector space V . Then

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$

Proof: For any $t \in \mathbb{C}$ let $\mathbf{v}_t = \mathbf{x} + t\mathbf{y}$. Then

$$\begin{aligned} \langle \mathbf{v}_t, \mathbf{v}_t \rangle &= \langle \mathbf{x} + t\mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} + t\mathbf{y} \rangle + t\langle \mathbf{y}, \mathbf{x} + t\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \bar{t}\langle \mathbf{x}, \mathbf{y} \rangle + t\langle \mathbf{y}, \mathbf{x} \rangle + t\bar{t}\langle \mathbf{y}, \mathbf{y} \rangle. \end{aligned}$$

Assume that $\mathbf{y} \neq \mathbf{0}$ and let $t = -\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$. Then

$$\langle \mathbf{v}_t, \mathbf{v}_t \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + t\langle \mathbf{y}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \frac{|\langle \mathbf{x}, \mathbf{y} \rangle|^2}{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Since $\langle \mathbf{v}_t, \mathbf{v}_t \rangle \geq 0$ the desired inequality follows. In the case $\mathbf{y} = \mathbf{0}$, we have $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = 0$.

Cauchy-Schwarz Inequality:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Corollary 1 $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Equivalently, for all $x_i, y_i \in \mathbb{R}$,

$$(x_1y_1 + \cdots + x_ny_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Corollary 2 For any $f, g \in C[a, b]$,

$$\left(\int_a^b f(x)g(x) dx \right)^2 \leq \int_a^b |f(x)|^2 dx \cdot \int_a^b |g(x)|^2 dx.$$

Norms induced by inner products

Theorem Suppose $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on a vector space V . Then $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is a norm.

Proof: Positivity is obvious. Homogeneity:

$$\|r\mathbf{x}\| = \sqrt{\langle r\mathbf{x}, r\mathbf{x} \rangle} = \sqrt{r\bar{r}\langle \mathbf{x}, \mathbf{x} \rangle} = |r| \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Triangle inequality (follows from Cauchy-Schwarz's):

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} + \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + 2\operatorname{Re}\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &\leq \langle \mathbf{x}, \mathbf{x} \rangle + 2|\langle \mathbf{x}, \mathbf{y} \rangle| + \langle \mathbf{y}, \mathbf{y} \rangle \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\| \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.\end{aligned}$$

Examples. • The length of a vector in \mathbb{R}^n ,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

is the norm induced by the dot product

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

• The norm $\|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$ on the vector space $C[a, b]$ is induced by the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx.$$

Angle

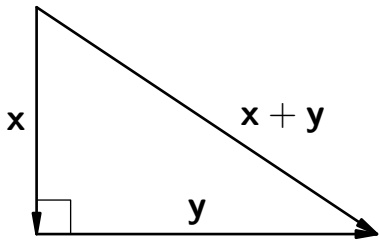
Since $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$, we can define the **angle** between nonzero vectors in any real vector space with an inner product (and induced norm):

$$\angle(\mathbf{x}, \mathbf{y}) = \arccos \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}.$$

Then $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos \angle(\mathbf{x}, \mathbf{y})$.

In particular, vectors \mathbf{x} and \mathbf{y} are **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

In a complex inner product space the angle between vectors is not defined. However the notion of orthogonality still makes sense.

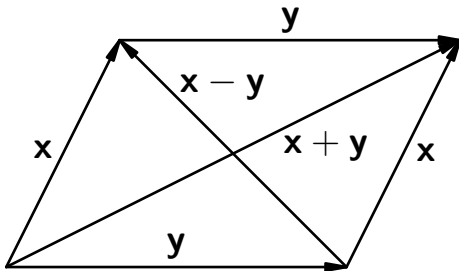


Pythagorean Law:

$$\mathbf{x} \perp \mathbf{y} \implies \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Proof:

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.\end{aligned}$$



Parallelogram Identity:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

Proof: $\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$.

Similarly, $\|\mathbf{x} - \mathbf{y}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle$.

Then $\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{y}, \mathbf{y} \rangle = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$.

Theorem 1 A norm on a vector space is induced by an inner product if and only if the Parallelogram Identity holds for this norm.

Theorem 2 (Polarization Identity) Suppose V is an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$.

(i) If V is a real vector space, then for any $\mathbf{x}, \mathbf{y} \in V$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4} (\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2).$$

(ii) If V is a complex vector space, then for any $\mathbf{x}, \mathbf{y} \in V$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4} (\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 + i\|\mathbf{x} + i\mathbf{y}\|^2 - i\|\mathbf{x} - i\mathbf{y}\|^2).$$