MATH 423

Linear Algebra II

Lecture 30:
The Gram-Schmidt process.
Orthogonal complement.

Orthogonal sets

Let V be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

Definition. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ form an **orthogonal set** if they are orthogonal to each other: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$.

If, in addition, all vectors are of unit norm, $\|\mathbf{v}_i\| = 1$, then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is called an **orthonormal set**.

Theorem Any orthogonal set of nonzero vectors is linearly independent.

Orthogonal basis

Theorem If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for an inner product space V, then

$$\mathbf{x} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n$$

for any vector $\mathbf{x} \in V$.

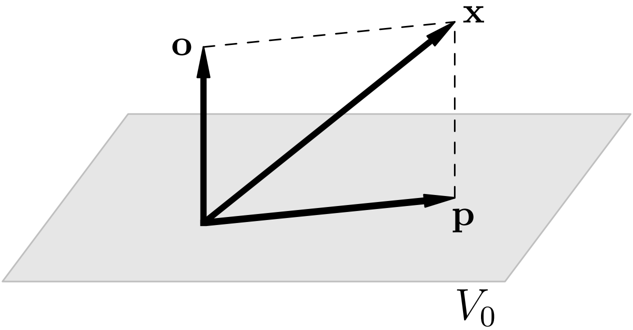
Corollary If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthonormal basis for an inner product space V, then

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{x}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle \mathbf{v}_n$$

for any vector $\mathbf{x} \in V$.

Orthogonal projection

Theorem Let V be an inner product space and V_0 be a finite-dimensional subspace of V. Then any vector $\mathbf{x} \in V$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V_0$ and $\mathbf{o} \perp V_0$.



The component $\mathbf{p}$ is called the **orthogonal projection** of the vector $\mathbf{x}$ onto the subspace V_0.

Theorem Let V be an inner product space and V_0 be a finite-dimensional subspace of V. Then any vector $\mathbf{x} \in V$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V_0$ and $\mathbf{o} \perp V_0$.

Proof of uniqueness: Suppose $\mathbf{x} = \mathbf{p} + \mathbf{o} = \mathbf{p}' + \mathbf{o}'$, where $\mathbf{p}, \mathbf{p}' \in V_0$, $\mathbf{o} \perp V_0$, $\mathbf{o}' \perp V_0$. Then $\mathbf{o} - \mathbf{o}' = \mathbf{p}' - \mathbf{p} \in V_0$. It follows that $\langle \mathbf{o}, \mathbf{o} - \mathbf{o}' \rangle = \langle \mathbf{o}', \mathbf{o} - \mathbf{o}' \rangle = 0$. Hence $\langle \mathbf{o} - \mathbf{o}', \mathbf{o} - \mathbf{o}' \rangle = \langle \mathbf{o}, \mathbf{o} - \mathbf{o}' \rangle - \langle \mathbf{o}', \mathbf{o} - \mathbf{o}' \rangle = 0$ so that $\mathbf{o} - \mathbf{o}' = \mathbf{0}$. Thus $\mathbf{o} = \mathbf{o}'$, then $\mathbf{p} = \mathbf{p}'$.

Proof of existence in the case V_0 admits an orthogonal basis: Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V_0. Let

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n$$

and $\mathbf{o} = \mathbf{x} - \mathbf{p}$. By construction, $\mathbf{x} = \mathbf{p} + \mathbf{o}$ and $\mathbf{p} \in V_0$. Just as in the previous lecture, we obtain that $\langle \mathbf{p}, \mathbf{v}_i \rangle = \langle \mathbf{x}, \mathbf{v}_i \rangle$ for $1 \leq i \leq n$. Then $\mathbf{o} \perp \mathbf{v}_i$ for all i, which implies that $\mathbf{o} \perp V_0$.

The Gram-Schmidt orthogonalization process

Let V be a vector space with an inner product. Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is a basis for V. Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

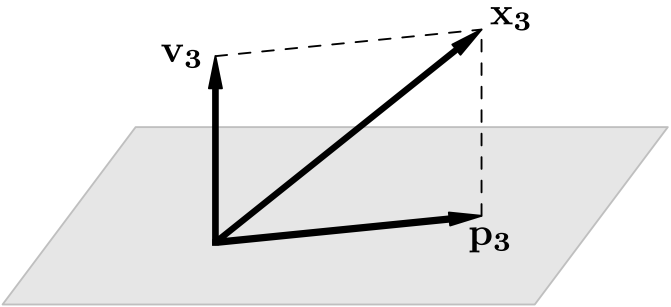
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2,$$

....

$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle}\mathbf{v}_{n-1}.$$

Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal basis for V.



$\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) = \mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2)$

$$\mathbf{p}_3 = \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

Any basis		Orthogonal basis
$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$	$\longrightarrow$	$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

Properties of the Gram-Schmidt process:

- $\mathbf{v}_k = \mathbf{x}_k - (\alpha_1 \mathbf{x}_1 + \cdots + \alpha_{k-1}\mathbf{x}_{k-1})$, $1 \leq k \leq n$;
- the span of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the same as the span of $\mathbf{x}_1, \ldots, \mathbf{x}_k$;
- $\mathbf{v}_k$ is orthogonal to $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$;
- $\mathbf{v}_k = \mathbf{x}_k - \mathbf{p}_k$, where $\mathbf{p}_k$ is the orthogonal projection of the vector $\mathbf{x}_k$ on the subspace spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$.

Normalization

Let V be a vector space with an inner product. Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V.

Let $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \dfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$,..., $\mathbf{w}_n = \dfrac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$.

Then $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ is an orthonormal basis for V.

Theorem Any finite-dimensional vector space with an inner product has an orthonormal basis.

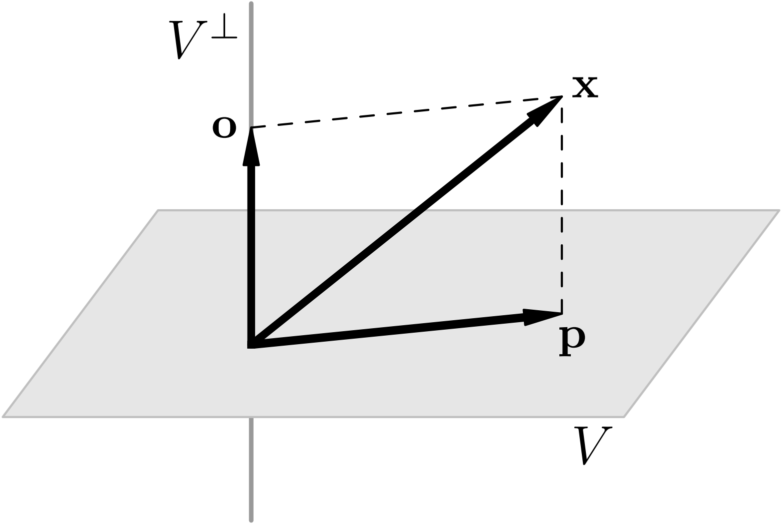
Remark. An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.

Orthogonal complement

Definition. Let S be a nonempty subset of an inner product space W. The **orthogonal complement** of S, denoted $S^\perp$, is the set of all vectors $\mathbf{x} \in W$ that are orthogonal to S.

Theorem Let V be a subspace of W. Then
(i) $V^\perp$ is a closed subspace of W;
(ii) $V \subset (V^\perp)^\perp$;
(iii) $V \cap V^\perp = \{\mathbf{0}\}$;
(iv) $\dim V + \dim V^\perp = \dim W$ if $\dim W < \infty$;
(v) if $\dim V < \infty$, then $V \oplus V^\perp = W$, that is, any vector $\mathbf{x} \in W$ is (uniquely) represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o} \in V^\perp$.

Remark. The orthogonal projection onto a subspace V is well defined if and only if $V \oplus V^\perp = W$.



V⊥
x
o
p
V

Suppose V is a subspace of an inner product space W such that $V \oplus V^{\perp} = W$. Let $\mathbf{p}$ be the orthogonal projection of a vector $\mathbf{x} \in W$ onto V.

Theorem $\|\mathbf{x} - \mathbf{v}\| > \|\mathbf{x} - \mathbf{p}\|$ for any $\mathbf{v} \neq \mathbf{p}$ in V.

Proof: Let $\mathbf{o} = \mathbf{x} - \mathbf{p}$, $\mathbf{o}_1 = \mathbf{x} - \mathbf{v}$, and $\mathbf{v}_1 = \mathbf{p} - \mathbf{v}$. Then $\mathbf{o}_1 = \mathbf{o} + \mathbf{v}_1$, $\mathbf{v}_1 \in V$, and $\mathbf{v}_1 \neq \mathbf{0}$. Since $\mathbf{o} \perp V$, it follows that $\langle \mathbf{o}, \mathbf{v}_1 \rangle = 0$.

$$\|\mathbf{o}_1\|^2 = \langle \mathbf{o}_1, \mathbf{o}_1 \rangle = \langle \mathbf{o} + \mathbf{v}_1, \mathbf{o} + \mathbf{v}_1 \rangle$$
$$= \langle \mathbf{o}, \mathbf{o} \rangle + \langle \mathbf{v}_1, \mathbf{o} \rangle + \langle \mathbf{o}, \mathbf{v}_1 \rangle + \langle \mathbf{v}_1, \mathbf{v}_1 \rangle$$
$$= \langle \mathbf{o}, \mathbf{o} \rangle + \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \|\mathbf{o}\|^2 + \|\mathbf{v}_1\|^2 > \|\mathbf{o}\|^2.$$

Thus $\|\mathbf{x} - \mathbf{p}\| = \min_{\mathbf{v} \in V} \|\mathbf{x} - \mathbf{v}\|$ is the **distance** from the vector $\mathbf{x}$ to the subspace V.

Problem. Find the distance from the point $\mathbf{y} = (0, 0, 0, 1)$ to the subspace $V \subset \mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, -1, 1, -1)$, $\mathbf{x}_2 = (1, 1, 3, -1)$, and $\mathbf{x}_3 = (-3, 7, 1, 3)$.

Let us apply the Gram-Schmidt process to vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}$. We should obtain an orthogonal set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. The desired distance will be $|\mathbf{v}_4|$.

$$\mathbf{x}_1 = (1, -1, 1, -1), \ \mathbf{x}_2 = (1, 1, 3, -1),$$
$$\mathbf{x}_3 = (-3, 7, 1, 3), \ \mathbf{y} = (0, 0, 0, 1).$$

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, -1, 1, -1),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = (1, 1, 3, -1) - \frac{4}{4}(1, -1, 1, -1)$$
$$= (0, 2, 2, 0),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$
$$= (-3, 7, 1, 3) - \frac{-12}{4}(1, -1, 1, -1) - \frac{16}{8}(0, 2, 2, 0)$$
$$= (0, 0, 0, 0).$$

The Gram-Schmidt process can be used to check linear independence of vectors!

The vector $\mathbf{x}_3$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.
V is a plane, not a 3-dimensional subspace.
We should orthogonalize vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}$.

$$\tilde{\mathbf{v}}_3 = \mathbf{y} - \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2$$

$$= (0,0,0,1) - \frac{-1}{4}(1,-1,1,-1) - \frac{0}{8}(0,2,2,0)$$

$$= (1/4, -1/4, 1/4, 3/4).$$

$$|\tilde{\mathbf{v}}_3| = \left|\left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right)\right| = \frac{1}{4}\,|(1,-1,1,3)| = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}.$$