MATH 423
Linear Algebra II

Lecture 31:
Dual space.
Adjoint operator.

Dual space

Let V be a vector space over a field $\mathbb{F}$.

Definition. The vector space $\mathcal{L}(V,\mathbb{F})$ of all linear functionals $\ell : V \to \mathbb{F}$ is called the **dual space** of V (denoted V' or V^*).

Theorem Let $\beta = \{\mathbf{v}_\alpha\}_{\alpha\in\mathcal{A}}$ be a basis for V. Then
(i) any linear functional $\ell : V \to \mathbb{F}$ is uniquely determined by its restriction to β;
(ii) any function $f : \beta \to \mathbb{F}$ can be (uniquely) extended to a linear functional on V.

Thus we have a one-to-one correspondence between elements of the dual space V' and collections of scalars c_α, $\alpha \in \mathcal{A}$. Namely, $\ell \mapsto \{\ell(\mathbf{v}_\alpha)\}_{\alpha\in\mathcal{A}}$.

Dual basis

Let $\beta = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$ be a basis for a vector space V. For any $1 \le i \le n$ let f_i denote a unique linear functional on V such that $f_i(\mathbf{v}_j) = 1$ if $i = j$ and 0 otherwise.

If $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, then $f_i(\mathbf{v}) = r_i$. That is, the functional f_i evaluates the ith coordinate of the vector $\mathbf{v}$ relative to the basis β.

Theorem The functionals $f_1, f_2, \dots, f_n$ form a basis for the dual space V' (called the **dual basis** of β).

Double dual space

The **double dual** of a vector space V is V'', the dual of V'.

Since V' is a functional vector space, to any vector $v \in V$ we associate an evaluation mapping, denoted $\hat{\mathbf{v}}$, given by $\hat{\mathbf{v}}(f) = f(\mathbf{v})$, $\mathbf{v} \in V$. This mapping is linear, hence it is an element of V''.

Theorem Consider a mapping $\chi : V \to V''$ given by $\chi(\mathbf{v}) = \hat{\mathbf{v}}$. Then
(i) χ is linear;
(ii) χ is one-to-one;
(iii) χ is onto if and only if $\dim V < \infty$.

Corollary 1 If V is finite-dimensional, then χ is an isomorphism of V onto V''.

Corollary 2 If V is finite-dimensional, then any basis for V' is the dual basis of some basis for V.

Dual linear transformation

Suppose V and W are vector spaces and $L : V \to W$ is a linear transformation. The **dual transformation** of L is a transformation $L' : W' \to V'$ given by $L'(f) = f \circ L$. That is, L' precomposes each linear functional on W with L.

It is easy to see that $L'(f)$ is indeed a linear functional on V. Also, L' is linear.

Suppose V and W are finite-dimensional. Let β be a basis for V and γ be a basis for W. Let β' be the dual basis of β and γ' be the dual basis for γ.

Theorem If $[L]_\beta^\gamma = A$ then $[L']_{\gamma'}^{\beta'} = A^t$.

Dual of an inner product space

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$. For any $\mathbf{y} \in V$ consider a function $\ell_{\mathbf{y}} : V \to \mathbb{F}$ given by $\ell_{\mathbf{y}}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x} \in V$. This function is linear.

Theorem Let $\theta : V \to V'$ be given by $\theta(\mathbf{v}) = \ell_{\mathbf{v}}$. Then **(i)** θ is linear if $\mathbb{F} = \mathbb{R}$ and half-linear if $\mathbb{F} = \mathbb{C}$; **(ii)** θ is one-to-one.

Corollary If V is finite-dimensional, then any linear functional on V is uniquely represented as $\ell_{\mathbf{v}}$ for some $\mathbf{v} \in V$.

Adjoint operator

Let L be a linear operator on an inner product space V.

Definition. The **adjoint** of L is a transformation $L^*: V \to V$ satisfying $\langle L(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, L^*(\mathbf{y}) \rangle$ for all $\mathbf{x}, \mathbf{y} \in V$.

Notice that the adjoint of L may not exist.

Theorem 1 If the adjoint L^* exists, then it is unique and linear.

Theorem 2 If V is finite-dimensional, then the adjoint operator L^* always exists.

Properties of adjoint operators:

- $(L_1 + L_2)^* = L_1^* + L_2^*$
- $(rL)^* = \overline{r}\, L^*$
- $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$
- $(L^*)^* = L$
- $\mathrm{id}_V^* = \mathrm{id}_V$