MATH 423

Linear Algebra II

# Lecture 32: Adjoint operator (continued). Normal operators.

# Dual of an inner product space

Let $V$ be a vector space with an inner product $\langle \cdot, \cdot \rangle$. For any $\mathbf{y} \in V$ consider a function $\ell_{\mathbf{y}} : V \to \mathbb{F}$ given by $\ell_{\mathbf{y}}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x} \in V$. This function is linear.

**Theorem** Let $\theta : V \to V'$ be given by $\theta(\mathbf{v}) = \ell_{\mathbf{v}}$. Then **(i)** $\theta$ is linear if $\mathbb{F} = \mathbb{R}$ and half-linear if $\mathbb{F} = \mathbb{C}$.

**(ii)** $\theta$ is one-to-one, that is, $\mathbf{v}$ is uniquely recovered by $\ell_{\mathbf{v}}$.

**(iii)** If $V$ is finite-dimensional, then $\theta$ is onto, i.e., any linear functional on $V$ is uniquely represented as $\ell_{\mathbf{v}}$ for some $\mathbf{v} \in V$.

# Adjoint operator

Let $L$ be a linear operator on an inner product space $V$.

*Definition.* The **adjoint** of $L$ is a transformation $L^*: V \to V$ satisfying $\langle L(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, L^*(\mathbf{y}) \rangle$ for all $\mathbf{x}, \mathbf{y} \in V$.

An equivalent condition is that $\ell_{\mathbf{y}} \circ L = \ell_{L^*(\mathbf{y})}$ for all $\mathbf{y} \in V$. Notice that the adjoint of $L$ may not exist.

**Theorem (i)** If the adjoint operator $L^*$ exists, it is unique and linear. **(ii)** If $V$ is finite-dimensional, then $L^*$ always exists.

*Properties of adjoint operators:*

- $(L_1 + L_2)^* = L_1^* + L_2^*$
- $(rL)^* = \overline{r}\, L^*$
- $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$
- $(L^*)^* = L$
- $\mathrm{id}_V^* = \mathrm{id}_V$

# Adjoint matrix

Suppose $A = (a_{ij})$ is an $m \times n$ matrix with complex entries. The **adjoint matrix** of $A$ is an $n \times m$ matrix $A^* = (b_{ij})$ such that $b_{ij} = \overline{a_{ji}}$. In other words, $A^* = \overline{A^t}$.

*Properties of adjoint matrices:*

- $(A + B)^* = A^* + B^*$
- $(rA)^* = \overline{r}\, A^*$
- $(AB)^* = B^* A^*$
- $(A^*)^* = A$
- $I^* = I$
- $(A^{-1})^* = (A^*)^{-1}$

**Theorem** Let $L$ be a linear operator on an inner product space $V$ of finite dimension. If $\beta$ is an orthonormal basis for $V$, then $[L^*]_\beta = ([L]_\beta)^*$.

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*Proof:* Let $\beta = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$. Let $A = (a_{ij})$ be the matrix of $L$ and $B = (b_{ij})$ be the matrix of $L^*$ relative to this basis.

By definition, $a_{ij}$ is the $i$th coordinate of the vector $L(\mathbf{v}_j)$. Since the basis $\beta$ is orthonormal, we have $a_{ij} = \langle L(\mathbf{v}_j), \mathbf{v}_i \rangle$. Likewise, $b_{ij} = \langle L^*(\mathbf{v}_j), \mathbf{v}_i \rangle$.

For any indices $i, j$,

$$b_{ij} = \langle L^*(\mathbf{v}_j), \mathbf{v}_i \rangle = \overline{\langle \mathbf{v}_i, L^*(\mathbf{v}_j) \rangle} = \overline{\langle L(\mathbf{v}_i), \mathbf{v}_j \rangle} = \overline{a_{ji}}.$$

Thus $B = A^*$.

*Example.* $V = \mathbb{C}^2$, $\langle \mathbf{x}, \mathbf{y} \rangle = x_1\overline{y_1} + x_2\overline{y_2}$.
$L(z_1, z_2) = (z_1 - 2iz_2,\ 3z_1 + iz_2)$.

$L$ is a linear operator. The matrix of $L$ relative to the standard basis is $A = \begin{pmatrix} 1 & -2i \\ 3 & i \end{pmatrix}$.

Since the standard basis is orthonormal, the matrix of the adjoint $L^*$ is $A^* = \overline{A^t} = \begin{pmatrix} 1 & 3 \\ 2i & -i \end{pmatrix}$.

Therefore $L^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2i & -i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

Equivalently, $L^*(z_1, z_2) = (z_1 + 3z_2,\ 2iz_1 - iz_2)$.

*Example.* $V = C^\infty([a,b])$, $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$.
$L(f) = f'$.

$$\langle L(f), g\rangle = \int_a^b f'(x)g(x)\,dx$$

$$= f(x)g(x)\Big|_{x=a}^{b} - \int_a^b f(x)g'(x)\,dx$$

$$= f(b)g(b) - f(a)g(a) + \langle f, -L(g)\rangle.$$

If $g(a) \neq 0$ or $g(b) \neq 0$, then there is no function $h \in C^\infty([a,b])$ such that

$$f(b)g(b) - f(a)g(a) = \langle f, h\rangle$$

for all $f \in C^\infty([a,b])$. Therefore the operator $L$ has no adjoint.

*Example.* $V = (C[a,b], \mathbb{C})$, $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$.

$(Lf)(x) = \int_a^b K(x,y)f(y)\,dy$, where $K$ is a continuous function on $[a,b]\times[a,b]$. The operator $L$ is called an **integral operator**; the function $K$ is called the **kernel** of $L$.

$$\langle L(f), g\rangle = \int_a^b \left(\int_a^b K(x,y)f(y)\,dy\right)\overline{g(x)}\,dx$$

$$= \int_a^b\int_a^b K(x,y)f(y)\overline{g(x)}\,dx\,dy$$

$$= \int_a^b f(y)\overline{\left(\int_a^b \overline{K(x,y)}\,g(x)\,dx\right)}\,dy = \langle f, \widetilde{L}(g)\rangle,$$

where $\widetilde{L}$ is an integral operator with the kernel $\widetilde{K}(x,y) = \overline{K(y,x)}$. Thus $\widetilde{L}$ is the adjoint operator of $L$.

# Normal operators

*Definition.* A linear operator $L$ on an inner product space $V$ is called **normal** if it commutes with its adjoint. That is, if the adjoint operator $L^*$ exists and $L \circ L^* = L^* \circ L$.

There are several special classes of normal operators important for applications.

The operator $L$ is **self-adjoint** if $L^* = L$.
Equivalently, $\langle L(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, L(\mathbf{y}) \rangle$ for all $\mathbf{x}, \mathbf{y} \in V$.

The operator $L$ is **anti-selfadjoint** if $L^* = -L$.
The operator $L$ is **unitary** if $L^* = L^{-1}$.

# Normal matrices

*Definition.* A square matrix $A$ with real or complex entries is **normal** if $AA^* = A^*A$.

**Theorem** Let $L$ be a linear operator on a finite-dimensional inner product space. Suppose $A$ is the matrix of $L$ relative to an orthonormal basis. Then the operator $L$ is normal if and only if the matrix $A$ is normal.

Special classes of normal operators give rise to special classes of normal matrices.

A matrix $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is **Hermitian** if $A^* = A$, **skew-Hermitian** if $A^* = -A$, and **unitary** if $A^* = A^{-1}$.

A square matrix $B$ with real entries is **symmetric** if $B^t = B$, **skew-symmetric** if $B^t = -B$, and **orthogonal** if $B^t = B^{-1}$.