MATH 423

Linear Algebra II

Lecture 33:
Diagonalization of normal operators.

Adjoint operator and adjoint matrix

Given a linear operator L on an inner product space V, the **adjoint** of L is a transformation $L^* : V \to V$ satisfying $\langle L(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, L^*(\mathbf{y}) \rangle$ for all $\mathbf{x}, \mathbf{y} \in V$.

Theorem 1 If V is finite-dimensional, then the adjoint operator L^* always exists.

Given a matrix A with complex entries, its **adjoint matrix** is $A^* = \overline{A^t}$.

Theorem 2 If A is the matrix of a linear operator L relative to an orthonormal basis β, then the matrix of L^* relative to the same basis is A^*.

Let $L : V \to V$ be a linear operator on an inner product space V. Recall that $\mathcal{N}(L)$ denotes the **null-space** of L and $\mathcal{R}(L)$ denotes the **range** of L:

$$\mathcal{N}(L) = \{\mathbf{x} \in V \mid L(\mathbf{x}) = \mathbf{0}\}, \quad \mathcal{R}(L) = \{L(\mathbf{y}) \mid \mathbf{y} \in V\}.$$

Theorem If the adjoint operator L^* exists, then $\mathcal{N}(L) = \mathcal{R}(L^*)^\perp$ as well as $\mathcal{N}(L^*) = \mathcal{R}(L)^\perp$.

Proof: $\mathbf{x} \in \mathcal{N}(L) \iff L(\mathbf{x}) = \mathbf{0} \iff \langle L(\mathbf{x}), \mathbf{y} \rangle = 0$ for all $\mathbf{y} \in V \iff \langle \mathbf{x}, L^*(\mathbf{y}) \rangle = 0$ for all $\mathbf{y} \in V \iff \mathbf{x} \perp \mathcal{R}(L^*)$ $\iff \mathbf{x} \in \mathcal{R}(L^*)^\perp$.

The second equality follows in the same way since $(L^*)^* = L$.

Example. $V = \mathbb{R}^n$ with the dot product, $L(\mathbf{x}) = A\mathbf{x}$, where $A \in \mathcal{M}_{n,n}(\mathbb{R})$ and vectors are regarded as column vectors.

We have $L^*(\mathbf{x}) = A^*\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The range of L^* is the column space of the matrix $A^* = A^t$, which is the row space of A. Therefore $\mathcal{N}(L) = \{\text{null-space of the matrix } A\}$ is the orthogonal complement of the row space of A.

Normal operators

Definition. A linear operator L on an inner product space V is called **normal** if it commutes with its adjoint. That is, if the adjoint operator L^* exists and $L \circ L^* = L^* \circ L$.

There are several special classes of normal operators important for applications.

The operator L is **self-adjoint** if $L^* = L$. Equivalently, $\langle L(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, L(\mathbf{y}) \rangle$ for all $\mathbf{x}, \mathbf{y} \in V$.

The operator L is **anti-selfadjoint** if $L^* = -L$.
The operator L is **unitary** if $L^* = L^{-1}$.

Normal matrices

Definition. A square matrix A with real or complex entries is **normal** if $AA^* = A^*A$.

Theorem Let L be a linear operator on a finite-dimensional inner product space. Suppose A is the matrix of L relative to an orthonormal basis. Then the operator L is normal if and only if the matrix A is normal.

Special classes of normal operators give rise to special classes of normal matrices.

A matrix $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is **Hermitian** if $A^* = A$, **skew-Hermitian** if $A^* = -A$, and **unitary** if $A^* = A^{-1}$.

A square matrix B with real entries is **symmetric** if $B^t = B$, **skew-symmetric** if $B^t = -B$, and **orthogonal** if $B^t = B^{-1}$.

Properties of normal operators

Theorem Suppose L is a normal operator on an inner product space V. Then

(i) $\|L(\mathbf{x})\| = \|L^*(\mathbf{x})\|$ for all $\mathbf{x} \in V$;

(ii) $\mathcal{N}(L) = \mathcal{N}(L^*)$;

(iii) an operator given by $\mathbf{x} \mapsto L(\mathbf{x}) - \lambda\mathbf{x}$ is normal for any scalar λ;

(iv) the operators L and L^* share eigenvectors; namely, if $L(\mathbf{v}) = \lambda\mathbf{v}$ then $L^*(\mathbf{v}) = \overline{\lambda}\mathbf{v}$;

(v) eigenvectors of L belonging to distinct eigenvalues are orthogonal;

(vi) if a subspace $V_0 \subset V$ is invariant under L, then the orthogonal complement $V_0^\perp$ is also invariant under L.

Properties of normal operators

- $\|L(\mathbf{x})\| = \|L^*(\mathbf{x})\|$ for all $\mathbf{x} \in V$.

$$\|L(\mathbf{x})\|^2 = \langle L(\mathbf{x}), L(\mathbf{x})\rangle = \langle \mathbf{x}, L^*(L(\mathbf{x}))\rangle = \langle \mathbf{x}, L(L^*(\mathbf{x}))\rangle$$
$$= \overline{\langle L(L^*(\mathbf{x})), \mathbf{x}\rangle} = \overline{\langle L^*(\mathbf{x}), L^*(\mathbf{x})\rangle} = \langle L^*(\mathbf{x}), L^*(\mathbf{x})\rangle = \|L(\mathbf{x})\|^2.$$

- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of L belonging to distinct eigenvalues λ_1 and λ_2, then $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$.

We have $L(\mathbf{v}_1) = \lambda_1\mathbf{v}_1$ and $L^*(\mathbf{v}_2) = \overline{\lambda_2}\mathbf{v}_2$. Then

$$\lambda_1\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \langle \lambda_1\mathbf{v}_1, \mathbf{v}_2\rangle = \langle L(\mathbf{v}_1), \mathbf{v}_2\rangle = \langle \mathbf{v}_1, L^*(\mathbf{v}_2)\rangle$$
$$= \langle \mathbf{v}_1, \overline{\lambda_2}\mathbf{v}_2\rangle = \lambda_2\langle \mathbf{v}_1, \mathbf{v}_2\rangle.$$

It follows that $(\lambda_1 - \lambda_2)\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$. Since $\lambda_1 \neq \lambda_2$, we obtain $\langle \mathbf{v}_1, \mathbf{v}_2\rangle = 0$.

Diagonalization of normal operators

Theorem A linear operator L on a finite-dimensional inner product space V is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of L.

Proof ("if"): Suppose β is an orthonormal basis consisting of eigenvectors of L. Then the matrix $A = [L]_\beta$ is diagonal, $A = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Since β is orthonormal, $[L^*]_\beta = A^* = \mathrm{diag}(\overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_n})$. Clearly, $AA^* = A^*A$. Hence $L \circ L^* = L^* \circ L$.

Idea of the proof ("only if"): The statement is derived from the following two lemmas.

Lemma 1 (Schur's Theorem) There exists an orthonormal basis β for V such that the matrix $[L]_\beta$ is upper triangular.

Lemma 2 If a normal matrix is upper triangular, then it is actually diagonal.

Diagonalization of normal operators

Theorem A linear operator L on a finite-dimensional inner product space V is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of L.

Corollary 1 Suppose L is a normal operator. Then
(i) L is self-adjoint if and only if all eigenvalues of L are real $(\overline{\lambda} = \lambda)$;
(ii) L is anti-selfadjoint if and only if all eigenvalues of L are purely imaginary $(\overline{\lambda} = -\lambda)$;
(iii) L is unitary if and only if all eigenvalues of L are of absolute value 1 $(\overline{\lambda} = \lambda^{-1})$.

Corollary 2 A linear operator L on a finite-dimensional, real inner product space V is self-adjoint if and only if there exists an orthonormal basis for V consisting of eigenvectors of L.

Diagonalization of normal matrices

Theorem **(a)** $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is normal $\iff$ there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A;
(b) $A \in \mathcal{M}_{n,n}(\mathbb{R})$ is symmetric $\iff$ there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Example. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$

- A is symmetric.
- A has three eigenvalues: 0, 2, and 3.
- Associated eigenvectors are $\mathbf{v}_1 = (-1, 0, 1)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (0, 1, 0)$, respectively.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1, \frac{1}{\sqrt{2}}\mathbf{v}_2, \mathbf{v}_3$ form an orthonormal basis for $\mathbb{R}^3$.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^t$
- A_ϕ is orthogonal
- $\det(A_\phi - \lambda I) = (\cos\phi - \lambda)^2 + \sin^2\phi$.
- Eigenvalues: $\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$, $\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.
- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$, $\mathbf{v}_2 = (1, i)$.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1$ and $\frac{1}{\sqrt{2}}\mathbf{v}_2$ form an orthonormal basis for $\mathbb{C}^2$.