MATH 423

Linear Algebra II

Lecture 34:
Unitary operators.
Orthogonal matrices.

Diagonalization of normal operators

Theorem A linear operator L on a finite-dimensional inner product space V is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of L.

Corollary 1 Suppose L is a normal operator. Then
(i) L is self-adjoint if and only if all eigenvalues of L are real $(\overline{\lambda} = \lambda)$;
(ii) L is anti-selfadjoint if and only if all eigenvalues of L are purely imaginary $(\overline{\lambda} = -\lambda)$;
(iii) L is unitary if and only if all eigenvalues of L are of absolute value 1 $(\overline{\lambda} = \lambda^{-1})$.

Idea of the proof: $L(\mathbf{x}) = \lambda\mathbf{x} \iff L^*(\mathbf{x}) = \overline{\lambda}\,\mathbf{x}$.

Corollary 2 A linear operator L on a finite-dimensional, real inner product space V is self-adjoint if and only if there exists an orthonormal basis for V consisting of eigenvectors of L.

Diagonalization of normal matrices

Theorem Matrix $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is normal if and only if there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A.

Corollary 1 Suppose $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is a normal matrix. Then
(i) A is Hermitian if and only if all eigenvalues of A are real;
(ii) A is skew-Hermitian if and only if all eigenvalues of A are purely imaginary;
(iii) A is unitary if and only if all eigenvalues of A are of absolute value 1.

Corollary 2 Matrix $A \in \mathcal{M}_{n,n}(\mathbb{R})$ is symmetric if and only if there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, $\phi \in \mathbb{R}$.

- $A_\phi A_\psi = A_{\phi+\psi}$

$$A_\phi A_\psi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\,\cos\psi - \sin\phi\,\sin\psi & -\cos\phi\,\sin\psi - \sin\phi\,\cos\psi \\ \sin\phi\,\cos\psi + \cos\phi\,\sin\psi & \cos\phi\,\cos\psi - \sin\phi\,\sin\psi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi+\psi) & -\sin(\phi+\psi) \\ \sin(\phi+\psi) & \cos(\phi+\psi) \end{pmatrix} = A_{\phi+\psi}.$$

- $A_0 = I$

$$A_0 = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, $\phi \in \mathbb{R}$.

- $A_\phi^{-1} = A_{-\phi}$

$A_\phi A_{-\phi} = A_{\phi+(-\phi)} = A_0 = I \implies A_\phi^{-1} = A_{-\phi}$.

- $A_{-\phi} = A_\phi^t$

$$A_{-\phi} = \begin{pmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = A_\phi^t.$$

- A_ϕ is orthogonal

$A_\phi^t = A_{-\phi} = A_\phi^{-1} \implies A_\phi$ is orthogonal.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, $\phi \in \mathbb{R}$.

Characteristic polynomial:

$$\det(A_\phi - \lambda) = \begin{vmatrix} \cos\phi - \lambda & -\sin\phi \\ \sin\phi & \cos\phi - \lambda \end{vmatrix} = (\cos\phi - \lambda)^2 + \sin^2\phi.$$

Eigenvalues: $\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$,
$\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.

Associated eigenvectors: $\mathbf{v}_1 = (1, -i)^t$, $\mathbf{v}_2 = (1, i)^t$.

$$A_\phi \mathbf{v}_1 = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos\phi + i\sin\phi \\ \sin\phi - i\cos\phi \end{pmatrix} = \lambda_1 \mathbf{v}_1.$$

Note that $\lambda_2 = \overline{\lambda_1}$ and $\mathbf{v}_2 = \overline{\mathbf{v}_1}$. Since the matrix A_ϕ has real entries, $A_\phi \mathbf{v}_1 = \lambda_1 \mathbf{v}_1$ implies $A_\phi \overline{\mathbf{v}_1} = \overline{\lambda_1}\, \overline{\mathbf{v}_1}$.

We have $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 1 \cdot 1 + (-i) \cdot \bar{i} = 1 + (-i)^2 = 0$,
$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 2$. Hence vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1$ and $\frac{1}{\sqrt{2}}\mathbf{v}_2$ form an orthonormal basis for $\mathbb{C}^2$.

Characterization of unitary matrices

Theorem Given an $n\times n$ matrix A with complex entries, the following conditions are equivalent:
(i) A is unitary: $A^* = A^{-1}$;
(ii) columns of A form an orthonormal basis for $\mathbb{C}^n$;
(iii) rows of A form an orthonormal basis for $\mathbb{C}^n$.

Sketch of the proof: Entries of the matrix A^*A are inner products of columns of A. Entries of AA^* are inner products of rows of A. It follows that $A^*A = I$ if and only if the columns of A form an orthonormal set. Similarly, $AA^* = I$ if and only if the rows of A form an orthonormal set.

The theorem implies that a unitary matrix is the transition matrix changing coordinates from one orthonormal basis to another.

Diagonalization of normal matrices: revisited

Theorem 1 Given an $n \times n$ matrix A with complex entries, the following conditions are equivalent:

(i) A is normal: $A^*A = AA^*$;
(ii) there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A;
(iii) there exists a diagonal matrix D and a unitary matrix U such that $A = UDU^{-1}$ $(= UDU^*)$.

Theorem 2 Given an $n \times n$ matrix A with real entries, the following conditions are equivalent:

(i) A is symmetric: $A^t = A$;
(ii) there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A;
(iii) there exists a diagonal matrix D (with real entries) and an orthogonal matrix U such that $A = UDU^{-1}$ $(= UDU^t)$.

Characterizations of unitary operators

Theorem Given a linear operator on a finite-dimensional inner product space V, the following conditions are equivalent:

(i) L is unitary;
(ii) $\langle L(\mathbf{x}), L(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y} \in V$;
(iii) $\|L(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in V$;
(iv) the matrix of A relative to an orthonormal basis is unitary;
(v) L maps some orthonormal basis for V to another orthonormal basis;
(vi) L maps any orthonormal basis for V to another orthonormal basis.

Proof that (i) $\Longrightarrow$ (ii): $\langle L(\mathbf{x}), L(\mathbf{y})\rangle = \langle \mathbf{x}, L^*(L(\mathbf{y}))\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$.