MATH 423

Linear Algebra II

# Lecture 35:
# Orthogonal matrices (continued).
# Rigid motions.
# Rotations in space.

# Characterization of unitary matrices

**Theorem** Given an $n \times n$ matrix $A$ with complex entries, the following conditions are equivalent:
**(i)** $A$ is unitary: $A^* = A^{-1}$;
**(ii)** columns of $A$ form an orthonormal basis for $\mathbb{C}^n$;
**(iii)** rows of $A$ form an orthonormal basis for $\mathbb{C}^n$.

The theorem implies that a unitary matrix is the transition matrix changing coordinates from one orthonormal basis to another.

## Unitary equivalence

*Definition.* Given matrices $A, B \in \mathcal{M}_{n,n}(\mathbb{C})$, we say that $A$ is **unitarily equivalent** to $B$ if $A = UBU^{-1}$ for some unitary matrix $U$.

Unitary equivalence is a special case of similarity. The matrices $A$ and $B$ are unitarily equivalent if they are matrices of the same linear operator on $\mathbb{C}^n$ with respect to two different orthonormal bases.

**Theorem** Matrix $A \in \mathcal{M}_{n,n}(\mathbb{C})$ is normal if and only if it is unitary equivalent to a diagonal matrix.

## Orthogonal equivalence

*Definition.* Given matrices $A, B \in \mathcal{M}_{n,n}(\mathbb{R})$, we say that $A$ is **orthogonally equivalent** to $B$ if $A = UBU^{-1}$ for some orthogonal matrix $U$.

Clearly, orthogonal equivalence implies unitary equivalence and similarity. The matrices $A$ and $B$ are orthogonally equivalent if they are matrices of the same linear operator on $\mathbb{R}^n$ with respect to two different orthonormal bases.

**Theorem** Matrix $A \in \mathcal{M}_{n,n}(\mathbb{R})$ is symmetric if and only if it is orthogonally equivalent to a diagonal matrix.

*Example.* $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, $\phi \in \mathbb{R}$.

- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^t$
- $A_\phi$ is orthogonal
- Eigenvalues: $\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$, $\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.
- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$, $\mathbf{v}_2 = (1, i)$.
- $\lambda_2 = \overline{\lambda_1}$ and $\mathbf{v}_2 = \overline{\mathbf{v}_1}$.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1$ and $\frac{1}{\sqrt{2}}\mathbf{v}_2$ form an orthonormal basis for $\mathbb{C}^2$.

# Rigid motions

*Definition.* A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** (or a **rigid motion**) if it preserves distances between points: $\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$.

*Examples.* • Translation: $f(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0$ is a fixed vector.

• Unitary linear operator: $f(\mathbf{x}) = A\mathbf{x}$, where $A$ is an orthogonal matrix.

• If $f_1$ and $f_2$ are two isometries, then the composition $f_2 \circ f_1$ is also an isometry.

**Theorem** Any isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and $A$ is an orthogonal matrix.

**Theorem** Any orthogonal matrix is orthogonally equivalent to a diagonal block matrix of the form

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or $-1$, and

$$R_j = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}, \;\; \phi_j \in \mathbb{R}.$$

*Remark.* Each rotation block $R_j$ corresponds to a pair of complex conjugate eigenvalues $e^{i\phi_j}$ and $e^{-i\phi_j}$ of the matrix.

*Classification of $2 \times 2$ orthogonal matrices (up to orthogonal equivalence):*

| | | |
|---|---|---|
| | $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| | rotation about the origin | reflection in a line |
| Determinant: | $1$ | $-1$ |
| Eigenvalues: | $e^{i\phi}$ and $e^{-i\phi}$ | $-1$ and $1$ |

*Classification of $3 \times 3$ orthogonal matrices (up to o.e.):*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

$A =$ rotation about a line; $B =$ reflection in a plane; $C =$ rotation about a line combined with reflection in the orthogonal plane.

$\det A = 1$, $\det B = \det C = -1$.

$A$ has eigenvalues $1$, $e^{i\phi}$, $e^{-i\phi}$. $B$ has eigenvalues $-1$, $1$, $1$. $C$ has eigenvalues $-1$, $e^{i\phi}$, $e^{-i\phi}$.

*Example.* Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ that acts on the standard basis as follows: $L(\mathbf{e}_1) = \mathbf{e}_2$, $L(\mathbf{e}_2) = \mathbf{e}_3$, $L(\mathbf{e}_3) = -\mathbf{e}_1$.

$L$ maps the standard basis to another orthonormal basis, which implies that $L$ is unitary, i.e., a rigid motion. The matrix of $L$ relative to the standard basis is $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

It is orthogonal, which is another proof that $L$ is unitary.

It follows from the classification that the operator $L$ is either a rotation about an axis, or a reflection in a plane, or the composition of a rotation about an axis with the reflection in the plane orthogonal to the axis.

$\det A = -1 < 0$ so that $L$ reverses orientation. Therefore $L$ is not a rotation. Further, $A^2 \neq I$ so that $L^2$ is not the identity map. Therefore $L$ is not a reflection.

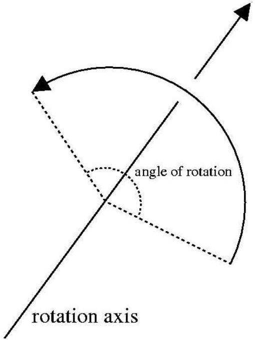
Since $L$ is a rotation about an axis composed with the reflection in the orthogonal plane, the matrix $A$ is orthogonally equivalent to

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix},$$

where $\phi$ is the angle of rotation. Orthogonally equivalent matrices are similar, and similar matrices have the same trace (since similar matrices have the same characteristic polynomial and the trace is one of its coefficients). Therefore $\mathrm{trace}(A) = -1 + 2\cos\phi$. On the other hand, $\mathrm{trace}(A) = 0$. Hence $-1 + 2\cos\phi = 0$. Then $\cos\phi = 1/2$ so that $\phi = 60^{\mathrm{o}}$.

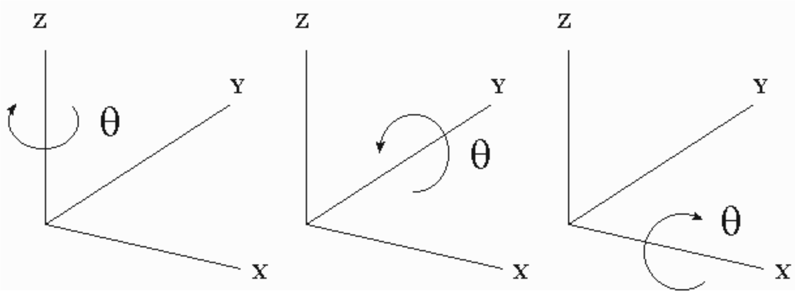
The axis of rotation consists of vectors $\mathbf{v}$ such that $A\mathbf{v} = -\mathbf{v}$. In other words, this is the eigenspace of $A$ associated to the eigenvalue $-1$. One can find that the eigenspace is spanned by the vector $(1, -1, 1)$.

# Rotations in space




If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).

# Clockwise rotations about coordinate axes




$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

**Problem.** Find the matrix of the rotation by $90^{\mathrm{o}}$ about the line spanned by the vector $\mathbf{a} = (1, 2, 2)$. The rotation is assumed to be counterclockwise when looking from the tip of **a**.

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ is the matrix of (counterclockwise) rotation by $90^{\mathrm{o}}$ about the $x$-axis.

We need to find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{v}_1$ points in the same direction as **a**. Also, the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ should obey the same hand rule as the standard basis. Then $B$ will be the matrix of the given rotation relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let $U$ denote the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (columns of $U$ are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$). Then the desired matrix is $A = UBU^{-1}$.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is going to be an orthonormal basis, the matrix $U$ will be orthogonal. Then $U^{-1} = U^t$ and $A = UBU^t$.

*Remark.* The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the same hand rule as the standard basis if and only if $\det U > 0$.

*Hint.* Vectors $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (-2, -1, 2)$, and $\mathbf{c} = (2, -2, 1)$ are orthogonal.

We have $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3$, hence $\mathbf{v}_1 = \frac{1}{3}\mathbf{a}$, $\mathbf{v}_2 = \frac{1}{3}\mathbf{b}$, $\mathbf{v}_3 = \frac{1}{3}\mathbf{c}$ is an orthonormal basis.

Transition matrix: $U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$.

$$\det U = \frac{1}{27}\begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.$$

(In the case $\det U = -1$, we would change $\mathbf{v}_3$ to $-\mathbf{v}_3$, or change $\mathbf{v}_2$ to $-\mathbf{v}_2$, or interchange $\mathbf{v}_2$ and $\mathbf{v}_3$.)

$$A = UBU^t$$

$$= \tfrac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \tfrac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \tfrac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \tfrac{1}{9}\begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}.$$