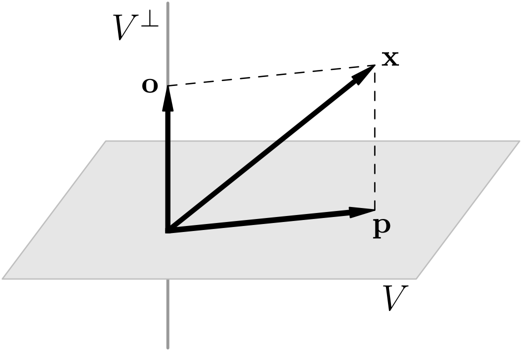
MATH 423

Linear Algebra II

Lecture 36:
Operator of orthogonal projection.

Operator of orthogonal projection

Let W be an inner product space and V be a subspace such that $V \oplus V^\perp = W$. Then we can define the operator P_V of **orthogonal projection** onto V. Namely, any vector $\mathbf{x} \in W$ is uniquely represented as $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o} \in V^\perp$, and we let $P_V(\mathbf{x}) = \mathbf{p}$.



Operator of orthogonal projection

Theorem 1 P_V is a linear operator.

Proof: Take any vectors $\mathbf{x}, \mathbf{y} \in W$. We have $\mathbf{x} = \mathbf{p}_1 + \mathbf{o}_1$ and $\mathbf{y} = \mathbf{p}_2 + \mathbf{o}_2$, where $\mathbf{p}_1, \mathbf{p}_2 \in V$ and $\mathbf{o}_1, \mathbf{o}_2 \in V^\perp$. Then

$$\mathbf{x} + \mathbf{y} = (\mathbf{p}_1 + \mathbf{p}_2) + (\mathbf{o}_1 + \mathbf{o}_2).$$

Since $\mathbf{p}_1 + \mathbf{p}_2 \in V$ and $\mathbf{o}_1 + \mathbf{o}_2 \in V^\perp$, it follows that $P_V(\mathbf{x} + \mathbf{y}) = \mathbf{p}_1 + \mathbf{p}_2 = P_V(\mathbf{x}) + P_V(\mathbf{y})$.

Further, for any scalar r we have $r\mathbf{x} = r\mathbf{p}_1 + r\mathbf{o}_1$. Since $r\mathbf{p}_1 \in V$ and $r\mathbf{o}_1 \in V^\perp$, we obtain $P_V(r\mathbf{x}) = r\mathbf{p}_1 = rP_V(\mathbf{x})$.

Thus P_V is a linear operator.

Operator of orthogonal projection

Theorem 2 (i) The range of P_V is V, the null-space is $V^\perp$.
(ii) P_V is idempotent, which means $P_V^2 = P_V$.
(iii) P_V is self-adjoint.

Proof: By definition of the operator P_V, it is zero when restricted to the subspace $V^\perp$ and the identity when restricted to the subspace V. This implies properties (i) and (ii).

Take any vectors $\mathbf{x}, \mathbf{y} \in W$. We have $\mathbf{x} = \mathbf{p}_1 + \mathbf{o}_1$, $\mathbf{y} = \mathbf{p}_2 + \mathbf{o}_2$, where $\mathbf{p}_1, \mathbf{p}_2 \in V$ and $\mathbf{o}_1, \mathbf{o}_2 \in V^\perp$. Then

$$\langle P_V(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{p}_1, \mathbf{p}_2 + \mathbf{o}_2\rangle = \langle \mathbf{p}_1, \mathbf{p}_2\rangle + \langle \mathbf{p}_1, \mathbf{o}_2\rangle = \langle \mathbf{p}_1, \mathbf{p}_2\rangle,$$

$$\langle \mathbf{x}, P_V(\mathbf{y})\rangle = \langle \mathbf{p}_1 + \mathbf{o}_1, \mathbf{p}_2\rangle = \langle \mathbf{p}_1, \mathbf{p}_2\rangle + \langle \mathbf{o}_1, \mathbf{p}_2\rangle = \langle \mathbf{p}_1, \mathbf{p}_2\rangle.$$

Thus $\langle P_V(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, P_V(\mathbf{y})\rangle$ so that P_V is self-adjoint.

Let L be a linear operator on an inner product space W.

Theorem 3 Suppose L is normal and idempotent: $L \circ L^* = L^* \circ L$ and $L^2 = L$. Then L is an operator of orthogonal projection.

Proof: Let V_0 and V_1 denote the eigenspaces of L associated with eigenvalues 0 and 1, respectively (if 0 or 1 is not an eigenvalue of L, the corresponding subspace is trivial). Since L is a normal operator, it follows that $V_0 \perp V_1$. In particular, $V_0 \cap V_1 = \{\mathbf{0}\}$, which implies that the sum of subspaces $V_0 + V_1$ is direct.

For any vector $\mathbf{x} \in W$ let $\mathbf{p} = L(\mathbf{x})$ and $\mathbf{o} = \mathbf{x} - \mathbf{p}$. Then

$$L(\mathbf{p}) = L(L(\mathbf{x})) = L^2(\mathbf{x}) = L(\mathbf{x}) = \mathbf{p},$$
$$L(\mathbf{o}) = L(\mathbf{x} - \mathbf{p}) = L(\mathbf{x}) - L(\mathbf{p}) = \mathbf{p} - \mathbf{p} = \mathbf{0}.$$

That is, $\mathbf{p} \in V_1$ and $\mathbf{o} \in V_0$. Therefore $V_1 \oplus V_0 = W$. Since $V_0 \perp V_1$, it follows that $V_0 = V_1^{\perp}$. Thus L is the operator of orthogonal projection onto the subspace V_1.

Example. $W = \mathbb{R}^3$, V is a plane spanned by vectors $\mathbf{x}_1 = (1,2,2)$ and $\mathbf{x}_2 = (0,6,3)$.

The operator of orthogonal projection onto V is given by

$$P_V(\mathbf{x}) = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2,$$

where $\mathbf{v}_1, \mathbf{v}_2$ is an arbitrary orthogonal basis for V. To get one, we apply the Gram-Schmidt process to the basis $\mathbf{x}_1, \mathbf{x}_2$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,2,2),$$
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = (0,6,3) - \frac{18}{9}(1,2,2) = (-2,2,-1).$$

Now for any vector $\mathbf{w} = (x,y,z) \in \mathbb{R}^3$ we obtain

$$P_V(\mathbf{w}) = \frac{x+2y+2z}{9}(1,2,2) + \frac{-2x+2y-z}{9}(-2,2,-1)$$
$$= \frac{1}{9}(5x-2y+4z,\ -2x+8y+2z,\ 4x+2y+5z).$$

Example. $W = \mathbb{R}^3$, V is the plane orthogonal to the vector $\mathbf{v} = (1, -2, 1)$.

By definition, $V = \{\mathbf{v}\}^\perp$. Therefore the orthogonal complement to V is spanned by $\mathbf{v}$. Hence the operator of orthogonal projection onto $V^\perp$ is given by $P_{V^\perp}(\mathbf{x}) = \dfrac{\langle \mathbf{x}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$.

Then the operator of orthogonal projection onto V is $P_V = \mathcal{I} - P_{V^\perp}$, where $\mathcal{I}$ is the identity map.

For any vector $\mathbf{w} = (x, y, z) \in \mathbb{R}^3$ we obtain

$$P_V(\mathbf{w}) = \mathbf{w} - \frac{\langle \mathbf{w}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} = (x, y, z) - \frac{x - 2y + z}{6}(1, -2, 1)$$

$$= \frac{1}{6}(5x + 2y - z,\ 2x + 2y + 2z,\ -x + 2y + 5z).$$

Matrix of P_V relative to the standard basis: $\dfrac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$.

Example. $W =$ the space of all 2π-periodic, piecewise continuous functions $f : \mathbb{R} \to \mathbb{C}$.

$V =$ the subspace spanned by $2n+1$ functions $h_{-n}, h_{-n+1}, \dots, h_{-1}, h_0, h_1, \dots, h_{n-1}, h_n$, where $h_k(x) = e^{ikx}$.

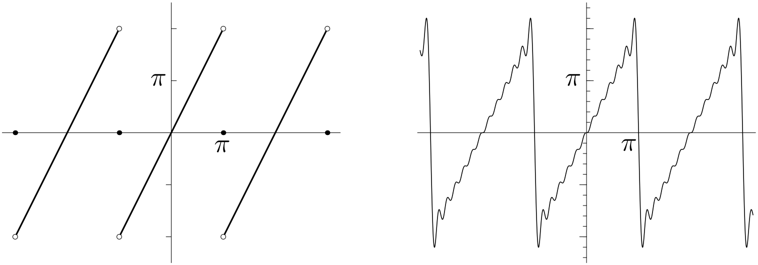
Inner product: $\langle f, g \rangle = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$.

The functions h_k form an orthonormal basis for V. The projection $g = P_V(f)$ is a partial sum of the **Fourier series** of the function f:

$$g(x) = \sum_{k=-n}^{n} c_k e^{ikx}, \quad \text{where} \quad c_k = \langle f, h_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky}\,dy.$$

It provides the best approximation of f by functions from V relative to the distance

$$\mathrm{dist}(f, g) = \|f - g\| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - g(x)|^2\,dx \right)^{1/2}.$$



Left graph: Function $f \in W$ such that $f(x) = 2x$ for $|x| < \pi$.

Right graph: Projection $P_V(f)$ in the case $n = 12$.