MATH 423

Linear Algebra II

Lecture 37:
Jordan blocks.
Jordan canonical form.

Jordan block

Definition. A **Jordan block** is an $n \times n$ matrix of the form

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

Examples. (λ), $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$.

The Jordan block of dimensions 2×2 or higher is the simplest example of a square matrix that is not diagonalizable.

Jordan block

Definition. A **Jordan block** is an $n \times n$ matrix of the form

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

Characteristic polynomial: $p(t) = \det(J - tI) = (\lambda - t)^n$.
Hence λ is the only eigenvalue.

It is easy to see that $J\mathbf{e}_1 = \lambda\mathbf{e}_1$ so that $\mathbf{e}_1 = (1, 0, \dots, 0)^t$ is an eigenvector. The consecutive columns of the matrix $J - \lambda I$ are $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n-1}$. It follows that $\mathrm{rank}(J - \lambda I) = n - 1$. Therefore the nullity of $J - \lambda I$ is 1. Thus the only eigenspace of the matrix J is the line spanned by $\mathbf{e}_1$.

Jordan canonical form

Definition. A square matrix B is in the **Jordan canonical form** if it has diagonal block structure

$$B = \begin{pmatrix} J_1 & O & \dots & O \\ O & J_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & J_k \end{pmatrix},$$

where each diagonal block J_i is a Jordan block.

The matrix B is called the **Jordan canonical form** (or **Jordan normal form**) of a square matrix A if A is similar to B, i.e., $A = UBU^{-1}$ for some invertible matrix U.

Note that a diagonal matrix is a special case of the Jordan canonical form.

Jordan canonical basis

Suppose B is a square matrix in the Jordan canonical form.

Given a linear operator $L : V \to V$ on a finite-dimensional vector space V, the matrix B is called the **Jordan canonical form** of L if B is the matrix of this operator relative to some basis β for V, $B = [L]_\beta$. The basis β is then called the **Jordan canonical basis** for L.

Let A be an $n \times n$ matrix and L_A denote an operator on $\mathbb{F}^n$ given by $L_A(\mathbf{x}) = A\mathbf{x}$. Then the Jordan canonical basis of L_A is called the Jordan canonical basis of A.

Note that a basis of eigenvectors is a special case of the Jordan canonical basis.

Let A be an $n \times n$ matrix such that the characteristic polynomial of A splits down to linear factors, i.e.,

$$\det(A - tI) = (\lambda_1 - t)(\lambda_2 - t) \dots (\lambda_n - t).$$

Theorem 1 Under the above assumption, the matrix A admits a Jordan canonical form.

Corollary If L is a linear operator on a finite-dimensional vector space such that the characteristic polynomial of L splits into linear factors, then L admits a Jordan canonical basis.

Theorem 2 Two matrices in Jordan canonical form are similar if and only if they coincide up to rearranging their Jordan blocks.

Corollary If a matrix or an operator admits a Jordan canonical form, then this form is unique up to rearranging the Jordan blocks.

Examples. $B_1 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix},$

$$B_2 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

All three matrices are in Jordan canonical form. Matrices B_1 and B_2 coincide up to rearranging their Jordan blocks. The matrix B_3 is essentially different.

Consider an $n \times n$ Jordan block

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

We already know that $J\mathbf{e}_1 = \lambda\mathbf{e}_1$ or, equivalently, $(J - \lambda I)\mathbf{e}_1 = \mathbf{0}$.

Then $(J - \lambda I)\mathbf{e}_2 = \mathbf{e}_1 \implies (J - \lambda I)^2\mathbf{e}_2 = \mathbf{0}$.

Next, $(J - \lambda I)\mathbf{e}_3 = \mathbf{e}_2 \implies (J - \lambda I)^3\mathbf{e}_3 = \mathbf{0}$.

In general, $(J - \lambda I)\mathbf{e}_k = \mathbf{e}_{k-1}$ and $(J - \lambda I)^k\mathbf{e}_k = \mathbf{0}$.

Hence multiplication by $J - \lambda I$ acts on the standard basis by a chain rule:

$$\overset{\mathbf{0}}{\circ} \longleftarrow \overset{\mathbf{e}_1}{\bullet} \longleftarrow \overset{\mathbf{e}_2}{\bullet} \longleftarrow \cdots \longleftarrow \overset{\mathbf{e}_{n-1}}{\bullet} \longleftarrow \overset{\mathbf{e}_n}{\bullet}$$

Example. $$B_1 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

Multiplication by $B_1 - 2I$:

$$\begin{array}{ccccc} \mathbf{0} & & \mathbf{e}_1 & & \mathbf{e}_2 \\ \circ & \longleftarrow & \bullet & \longleftarrow & \bullet \end{array}$$

Multiplication by $B_1 - 3I$:

$$\begin{array}{ccccccc} \mathbf{0} & & \mathbf{e}_3 & & & & \\ \circ & \longleftarrow & \bullet & & & & \\ \mathbf{0} & & \mathbf{e}_4 & & \mathbf{e}_5 & & \mathbf{e}_6 \\ \circ & \longleftarrow & \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet \end{array}$$