MATH 423

Linear Algebra II

Lecture 38: Generalized eigenvectors. Jordan canonical form (continued).

Jordan canonical form

A **Jordan block** is a square matrix of the form

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

A square matrix B is in the **Jordan canonical form** if it has diagonal block structure

$$B = \begin{pmatrix} J_1 & O & \dots & O \\ O & J_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & J_k \end{pmatrix},$$

where each diagonal block J_i is a Jordan block.

Consider an $n \times n$ Jordan block

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

Multiplication by $J - \lambda I$ acts on the standard basis by a chain rule:

$$\overset{\mathbf{0}}{\circ} \longleftarrow \overset{\mathbf{e}_1}{\bullet} \longleftarrow \overset{\mathbf{e}_2}{\bullet} \longleftarrow \cdots \longleftarrow \overset{\mathbf{e}_{n-1}}{\bullet} \longleftarrow \overset{\mathbf{e}_n}{\bullet}$$

Consider a matrix $B = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$.

This matrix is in Jordan canonical form. Vectors from the standard basis are organized in several chains.

Multiplication by $B - 2I$:

$$\overset{\mathbf{0}}{\circ} \longleftarrow \overset{\mathbf{e}_1}{\bullet} \longleftarrow \overset{\mathbf{e}_2}{\bullet}$$

Multiplication by $B - 3I$:

$$\overset{\mathbf{0}}{\circ} \longleftarrow \overset{\mathbf{e}_3}{\bullet}$$

$$\overset{\mathbf{0}}{\circ} \longleftarrow \overset{\mathbf{e}_4}{\bullet} \longleftarrow \overset{\mathbf{e}_5}{\bullet} \longleftarrow \overset{\mathbf{e}_6}{\bullet}$$

Generalized eigenvectors

Let $L : V \to V$ be a linear operator on a vector space V.

Definition. A nonzero vector $\mathbf{v}$ is called a **generalized eigenvector** of L associated with an eigenvalue λ if $(L - \lambda\mathcal{I})^k(\mathbf{v}) = \mathbf{0}$ for some integer $k \geq 1$ (here $\mathcal{I}$ denotes the identity map on V).

The set of all generalized eigenvectors for a particular λ along with the zero vector is called the **generalized eigenspace** and denoted K_λ.

The generalized eigenvectors and eigenspaces of an $n \times n$ matrix A are those of the operator $L_A(\mathbf{x}) = A\mathbf{x}$ on $\mathbb{F}^n$.

Properties of generalized eigenvectors

Let $L : V \to V$ be a linear operator on a vector space V.

Theorem (i) K_λ is a subspace of V containing the eigenspace $\mathcal{E}_\lambda$ and invariant under the operator L.

(ii) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are generalized eigenvectors associated with distinct eigenvalues, then they are linearly independent.

(iii) If $\dim V < \infty$ and the characteristic polynomial of L splits into linear factors, then $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_m}$, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of L.

Remarks. A Jordan canonical basis for the operator L (i.e., a basis β such that the matrix $[L]_\beta$ is in Jordan canonical form) must consist of generalized eigenvectors.

Not every basis of generalized eigenvectors is a Jordan canonical basis. However such a basis can always be transformed into a Jordan canonical basis.

Problem. Find the Jordan canonical form and the corresponding Jordan canonical basis for the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Characteristic polynomial:

$$p(\lambda) = \begin{vmatrix} -1-\lambda & 0 & 0 & 0 & 0 & 1 \\ 2 & -\lambda & -1 & -1 & 0 & -1 \\ 0 & 0 & 1-\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda & 0 \\ -1 & 0 & 0 & 0 & 0 & 1-\lambda \end{vmatrix}.$$

We expand the determinant by the 5th column, then by the 2nd column:

$$p(\lambda) = -\lambda(1-\lambda) \begin{vmatrix} -1-\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ -1 & 0 & 0 & 1-\lambda \end{vmatrix}.$$

Expand the determinant by the 3rd row, then by the 2nd column:

$$p(\lambda) = -\lambda(1-\lambda)^2 \begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix}$$

$$= -\lambda(1-\lambda)^3 \begin{vmatrix} -1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix}$$

$$= -\lambda(1-\lambda)^3 \big((-1-\lambda)(1-\lambda)+1\big)$$

$$= \lambda^3(\lambda-1)^3.$$

The eigenvalues are 0 and 1, both of multiplicity 3.

To find $\mathcal{N}(A)$, we convert A to reduced row echelon form:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathcal{N}(A)$ if $x_1 = x_3 = x_4 = x_5 = x_6 = 0$.

The eigenspace associated to the eigenvalue 0 is one-dimensional, spanned by $\mathbf{v}_1 = (0, 1, 0, 0, 0, 0)$.

The Jordan canonical form of A will contain only one Jordan block with the eigenvalue 0. Also, $\mathbf{v}_1$ can be extended to a chain of generalized eigenvectors in a Jordan canonical basis. The other two vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ in the chain should satisfy $A\mathbf{v}_2 = \mathbf{v}_1$ and $A\mathbf{v}_3 = \mathbf{v}_2$.

The equation $A\mathbf{x} = \mathbf{v}_1$ is equivalent to a system of scalar linear equations. General solution:

$$\mathbf{x} = (1, 0, 0, 0, 0, 1) + t(0, 1, 0, 0, 0, 0), \quad t \in \mathbb{R}.$$

We can take $\mathbf{v}_2 = (1, 0, 0, 0, 0, 1)$. Further, the general solution of the equation $A\mathbf{x} = \mathbf{v}_2$ is

$$\mathbf{x} = (1, 0, 0, 0, 0, 2) + t(0, 1, 0, 0, 0, 0), \quad t \in \mathbb{R}.$$

We can take $\mathbf{v}_3 = (1, 0, 0, 0, 0, 2)$.

To find the eigenspace $\mathcal{N}(A-I)$, we convert the matrix $A-I$ to reduced row echelon form:

$$A - I = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathcal{N}(A-I)$ if $x_1 = x_2 + x_3 = x_4 = x_6 = 0$.

We obtain that $\dim \mathcal{N}(A-I) = 2$. The eigenspace $\mathcal{N}(A-I)$ is spanned by $\mathbf{w}_1 = (0, -1, 1, 0, 0, 0)$ and $\mathbf{w}_2 = (0, 0, 0, 0, 1, 0)$.

Since $\dim \mathcal{N}(A - I) = 2$, the Jordan canonical form of A will contain two Jordan blocks with the eigenvalue 1. Since the multiplicity of 1 as a root of the characteristic polynomial of A is 3, there will be one block of dimensions 1×1 and one block of dimensions 2×2.

For the Jordan canonical basis, we need an eigenvector that can be extended to a chain of length 2. However one cannot guarantee that $\mathbf{w}_1$ or $\mathbf{w}_2$ is such a vector. We are going to build the 2-chain from the beginning rather than backwards. To this end, we find the null-space of the matrix $(A - I)^2$.

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & -2 \\ -5 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Row reduction: $(A-I)^2 = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & -2 \\ -5 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$

$$\to \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -5 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathcal{N}((A-I)^2)$ if $x_1 = x_2 + x_3 = x_6 = 0$.

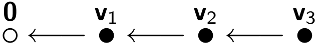
General solution of the equation $(A-I)^2\mathbf{x} = \mathbf{0}$:
$\mathbf{x} = t(0, -1, 1, 0, 0, 0) + s(0, 0, 0, 1, 0, 0) + r(0, 0, 0, 0, 1, 0)$.

We know that vectors $\mathbf{w}_1 = (0, -1, 1, 0, 0, 0)$ and $\mathbf{w}_2 = (0, 0, 0, 0, 1, 0)$ form a basis for $\mathcal{N}(A-I)$. The vector $\mathbf{w}_3 = (0, 0, 0, 1, 0, 0)$ extends it to a basis for $\mathcal{N}((A-I)^2)$.

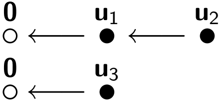
Since $\mathbf{w}_3 = (0,0,0,1,0,0)$ is in $\mathcal{N}((A-I)^2)$ but not in $\mathcal{N}(A-I)$, the vectors $\mathbf{u}_1 = (A-I)\mathbf{w}_3$ and $\mathbf{u}_2 = \mathbf{w}_3$ form a chain in a Jordan canonical basis for A. We obtain that $\mathbf{u}_1 = (0,-1,1,0,0,0) = \mathbf{w}_1$.

The last vector $\mathbf{u}_3$ for the Jordan canonical basis should extend the set $\{\mathbf{u}_1, \mathbf{u}_2\}$ to a basis for $\mathcal{N}((A-I)^2)$. We can take $\mathbf{u}_3 = \mathbf{w}_2 = (0,0,0,0,1,0)$.

Multiplication by A:



Multiplication by $A-I$:



Jordan canonical basis: $\mathbf{v}_1 = (0, 1, 0, 0, 0, 0)$, $\mathbf{v}_2 = (1, 0, 0, 0, 0, 1)$, $\mathbf{v}_3 = (1, 0, 0, 0, 0, 2)$, $\mathbf{u}_1 = (0, -1, 1, 0, 0, 0)$, $\mathbf{u}_2 = (0, 0, 0, 1, 0, 0)$, $\mathbf{u}_3 = (0, 0, 0, 0, 1, 0)$.

We have $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Here B is the Jordan canonical form of the matrix A while U is the transition matrix from the Jordan canonical basis to the standard basis.

Problem. Let $A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Find A^n.

Since $A = UBU^{-1}$, we have $A^n = UB^nU^{-1}$. Hence the problem is reduced to the computation of B^n. The matrix B has diagonal block structure:

$$B = \begin{pmatrix} J_1 & O & O \\ O & J_2 & O \\ O & O & J_3 \end{pmatrix} \implies B^n = \begin{pmatrix} J_1^n & O & O \\ O & J_2^n & O \\ O & O & J_3^n \end{pmatrix}.$$

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ J_1^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ J_1^3 = O.$$

$$J_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ J_2^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ J_2^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \ J_2^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Finally, $J_3 = (1) \implies J_3^n = (1)$.