MATH 423–200/500 April 13, 2012

Quiz 1: Solution

Problem. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1,2,2,0)$, $\mathbf{x}_2 = (1,-2,-3,2)$, and $\mathbf{x}_3 = (-1,0,5,-1)$.

(i) Find an orthogonal basis for V.

Let us apply the Gram-Schmidt orthogonalization process to the spanning set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. If this set is a basis for V (i.e., if the vectors are linearly independent), the process will yield an orthogonal basis for V. Otherwise the process will produce the zero vector at some point. We obtain

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,2,2,0), \qquad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1,-2,-3,2) - \frac{-9}{9}(1,2,2,0) = (2,0,-1,2),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (-1,0,5,-1) - \frac{9}{9}(1,2,2,0) - \frac{-9}{9}(2,0,-1,2) = (0,-2,2,1).$$

Thus $\mathbf{v}_1 = (1,2,2,0)$, $\mathbf{v}_2 = (2,0,-1,2)$, $\mathbf{v}_3 = (0,-2,2,1)$ is an orthogonal basis for V.

(ii) Find the distance from the point $\mathbf{y} = (0,9,0,9)$ to the subspace V.

The vector $\mathbf{y}$ is uniquely represented as $\mathbf{y} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o}$ is orthogonal to V. The vector $\mathbf{p}$ is the orthogonal projection of $\mathbf{y}$ onto the subspace V. Therefore the distance from the point $\mathbf{y}$ to V equals $\|\mathbf{y} - \mathbf{p}\| = \|\mathbf{o}\|$.

The orthogonal projection $\mathbf{p}$ of the vector $\mathbf{y}$ onto the subspace V is easily computed when we have an orthogonal basis for V. Using the orthogonal basis $\mathbf{v}_1 = (1,2,2,0)$, $\mathbf{v}_2 = (2,0,-1,2)$, $\mathbf{v}_3 = (0,-2,2,1)$ obtained earlier, we get

$$\mathbf{p} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{y} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 =$$

$$= \frac{18}{9}(1,2,2,0) + \frac{18}{9}(2,0,-1,2) + \frac{-9}{9}(0,-2,2,1) = (6,6,0,3).$$

Consequently, $\mathbf{o} = \mathbf{y} - \mathbf{p} = (0,9,0,9) - (6,6,0,3) = (-6,3,0,6)$. Thus the distance from $\mathbf{y}$ to the subspace V equals $\|\mathbf{o}\| = 9$.