MATH 423–200/500 April 20, 2012

Quiz 2: Solutions

Problem 1. Let A be a square matrix with real entries. Suppose that A is both skew-symmetric and orthogonal. Show that A has no eigenvalues other than i and $-i$. Section 200 students, also show that i and $-i$ are indeed eigenvalues of the matrix A.

Since the matrix A is skew-symmetric, all eigenvalues of A are purely imaginary. Since A is orthogonal, all eigenvalues are of absolute value 1. The only purely imaginary numbers of absolute value 1 are i and $-i$.

Any matrix with complex entries has at least one eigenvalue. By the above i or $-i$ is an eigenvalue of the matrix A. Since A has real entries, any nonreal eigenvalue λ of this matrix should be accompanied by the complex conjugate eigenvalue $\overline{\lambda}$. Therefore both i and $-i$ are eigenvalues of A.

Problem 2. Consider a linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L(x, y) = (x - 2y, 2x + y)$ for all $(x, y) \in \mathbb{R}^2$. Is L self-adjoint? Is L normal? Explain. (The inner product on $\mathbb{R}^2$ is the dot product.)

The matrix of the linear operator L relative to the standard basis is

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Clearly, A is not symmetric as $A \neq A^*$. On the other hand, one can easily check that $AA^* = A^*A = 5I$, in particular, A is normal. Since the standard basis is orthonormal, it follows that the operator L is normal, but not self-adjoint.