MATH 423–200/500 April 27, 2012

Quiz 3: Solution

Problem. Let R denote a linear operator on $\mathbb{R}^3$ that acts on vectors from the standard basis as follows: $R(\mathbf{e}_1) = \mathbf{e}_3$, $R(\mathbf{e}_2) = \mathbf{e}_1$, $R(\mathbf{e}_3) = \mathbf{e}_2$.

(i) Is R a rotation about an axis? Is R a reflection in a plane? Explain your answers.

(ii) If R is a rotation, find the axis and the angle. If R is a reflection, find the plane. If R is neither rotation nor reflection, describe the action of R in geometric terms.

The matrix of the operator R (relative to the standard basis) is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This matrix is orthogonal. Therefore R is a rigid motion. According to the classification, R is either a rotation about an axis, or a reflection in a plane, or the composition of two. Since $\det A = 1 > 0$, R is a rotation.

As R is a rotation about an axis, the matrix A is similar to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix},$$

where ϕ is the angle of rotation. Similar matrices have the same trace (since similar matrices have the same characteristic polynomial and the trace is one of its coefficients). The trace of A is 0. Hence $1 + 2\cos\phi = 0$. Then $\cos\phi = -1/2$ so that $\phi = 2\pi/3$.

The axis of the rotation R is the set of all points fixed by R. For any vector $(x, y, z) \in \mathbb{R}^3$ we have

$$R(x, y, z) = R(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) = xR(\mathbf{e}_1) + yR(\mathbf{e}_2) + zR(\mathbf{e}_3) = x\mathbf{e}_3 + y\mathbf{e}_1 + z\mathbf{e}_2 = (y, z, x).$$

Therefore $R(x, y, z) = (x, y, z)$ if and only if $x = y = z$. Thus the axis of the rotation is the line spanned by the vector $(1, 1, 1)$.