MATH 423-500/200 February 17, 2012

Test 1: Solutions

Problem 1 (20 pts.) Determine which of the following subsets of the vector space $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y - z = 0$.
(ii′) The set S_2' of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y - z = 0$ and $2y - 3z = 0$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x^2 - y^2 = 0$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $2y - 3z = 0$ and $2x - 3y - 1 = 0$.
(iv′) The set S_4' of vectors $(x, y, z) \in \mathbb{R}^3$ such that $e^x + e^z = 0$.

Solution: S_2 and S_2' are subspaces of $\mathbb{R}^3$, the other sets are not.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, a subspace must not be empty.

The set S_1 is the union of three planes $x = 0$, $y = 0$, and $z = 0$. It is not closed under addition as the following example shows: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$.

S_2 is a plane passing through the origin. It is easy to check that S_2 is closed under addition and scalar multiplication. Alternatively, S_2 is a subspace of $\mathbb{R}^3$ since it is the null-space of a linear functional $\ell : \mathbb{R}^3 \to \mathbb{R}$ given by $\ell(x, y, z) = x + y - z$, $(x, y, z) \in \mathbb{R}^3$.

S_2' is a subspace of $\mathbb{R}^3$ since it is the null-space of a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ 2y - 3z \end{pmatrix}$$

for all $x, y, z \in \mathbb{R}$.

Since $x^2 - y^2 = (x - y)(x + y)$, the set S_3 is the union of two planes $x - y = 0$ and $x + y = 0$. The following example shows that S_3 is not closed under addition: $(1, 1, 0) + (1, -1, 0) = (2, 0, 0)$.

The set S_4 is the intersection of two planes $2y - 3z = 0$ and $2x - 3y = 1$. Hence S_4 is a line. One of the planes does not pass through the origin so that S_4 does not contain the zero vector. Therefore this set is not a subspace.

Since $e^x > 0$ for any $x \in \mathbb{R}$, the set S_4' is empty. The empty set is not a subspace.

Thus S_2 and S_2' are subspaces of $\mathbb{R}^3$ while S_1, S_3, S_4, and S_4' are not.

Problem 2 (25 pts.) Let W be a subspace of $\mathcal{M}_{2,2}(\mathbb{R})$ spanned by matrices $A, A^2, A^3, \dots, A^n, \dots$, where

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Find a basis for W, then extend it to a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

Solution: $\{A, A^2\}$ is a basis for W; the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ extend it to a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

First we compute several powers of the matrix A:

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^4 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $A^3 = I$, we have $A^{k+3} = A^k A^3 = A^k$ for any integer $k > 0$. It follows that $A^{3m} = I$, $A^{1+3m} = A$, and $A^{2+3m} = A^2$ for any integer $m > 0$. Therefore the subspace W is spanned by the matrices A, A^2, and $A^3 = I$. Further, we have $A + A^2 + A^3 = O$. Hence $A^3 = -A - A^2$, which implies that A and A^2 span W as well. Clearly, A and A^2 are linearly independent. Therefore $\{A, A^2\}$ is a basis for W.

The matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for the vector space $\mathcal{M}_{2,2}(\mathbb{R})$. Therefore we can extend the set $\{A, A^2\}$ to a basis for $\mathcal{M}_{2,2}(\mathbb{R})$ by adding two of these matrices. For example, $\{A, A^2, E_1, E_2\}$ is a basis. To verify this, it is enough to show that the matrices A, A^2, E_1, E_2 are linearly independent. Assume that $r_1 A + r_2 A^2 + r_3 E_1 + r_4 E_2 = O$ for some scalars $r_1, r_2, r_3, r_4 \in \mathbb{R}$. Since

$$r_1 A + r_2 A^2 + r_3 E_1 + r_4 E_2 = r_1 \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} + r_3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + r_4 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -r_1 + r_3 & r_1 - r_2 + r_4 \\ -r_1 + r_2 & -r_2 \end{pmatrix},$$

we have $-r_1 + r_3 = r_1 - r_2 + r_4 = -r_1 + r_2 = -r_2 = 0$. It follows that $r_1 = r_2 = r_3 = r_4 = 0$. Thus A, A^2, E_1, E_2 are linearly independent.

Problem 3 (20 pts.) Let V_1, V_2, and V_3 be finite-dimensional vector spaces. Suppose that $L : V_1 \to V_2$ and $T : V_2 \to V_3$ are linear transformations. Prove that $\operatorname{rank}(T \circ L) \le \operatorname{rank}(L)$ and $\operatorname{rank}(T \circ L) \le \operatorname{rank}(T)$.

Since $(T \circ L)(\mathbf{x}) = T(L(\mathbf{x}))$ for any $\mathbf{x} \in V_1$, it follows that the range of the composition $T \circ L$ is contained in the range of T: $\mathcal{R}(T \circ L) \subset \mathcal{R}(T)$. Then $\dim \mathcal{R}(T \circ L) \le \dim \mathcal{R}(T)$, that is, $\operatorname{rank}(T \circ L) \le \operatorname{rank}(T)$.

By the Dimension Theorem, $\dim \mathcal{R}(L) + \dim \mathcal{N}(L) = \dim \mathcal{R}(T \circ L) + \dim \mathcal{N}(T \circ L) = \dim V_1$. Since $\operatorname{rank}(L) = \dim \mathcal{R}(L)$ and $\operatorname{rank}(T \circ L) = \dim \mathcal{R}(T \circ L)$, the inequality $\operatorname{rank}(T \circ L) \le \operatorname{rank}(L)$ is equivalent to the inequality $\dim \mathcal{N}(T \circ L) \ge \dim \mathcal{N}(L)$. We are going to prove the latter.

Let $\mathbf{0}_i$ denote the zero vector in the vector space V_i, $1 \le i \le 3$. If $L(\mathbf{x}) = \mathbf{0}_2$ for some vector $\mathbf{x} \in V_1$, then $(T \circ L)(\mathbf{x}) = T(L(\mathbf{x})) = T(\mathbf{0}_2)$, which equals $\mathbf{0}_3$ since the transformation T is linear. This means that the null-space of L is contained in the null-space of $T \circ L$: $\mathcal{N}(L) \subset \mathcal{N}(T \circ L)$. Consequently, $\dim \mathcal{N}(L) \le \dim \mathcal{N}(T \circ L)$.

Problem 4 (25 pts.) The functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$ span a 4-dimensional subspace V of the vector space $\mathcal{F}(\mathbb{R})$. Consider a linear transformation $D : V \to \mathcal{F}(\mathbb{R})$ given by $D(f) = f'$ for all functions $f \in V$.

(i) Show that the range of D is V and the null-space of D is trivial.

(ii) Find the matrix of D (regarded as an operator on V) relative to the basis f_1, f_2, f_3, f_4.

Solution: the matrix of D is $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

Since it is given that the functions f_1, f_2, f_3, f_4 span a 4-dimensional subspace, they must be linearly independent and form a basis for the subspace. First we compute the images of these functions under the transformation D:

$$\begin{aligned}
(Df_1)(x) &= f_1'(x) = (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x), \\
(Df_2)(x) &= f_2'(x) = (x \cos x)' = -x \sin x + \cos x = -f_1(x) + f_4(x), \\
(Df_3)(x) &= f_3'(x) = (\sin x)' = \cos x = f_4(x), \\
(Df_4)(x) &= f_4'(x) = (\cos x)' = -\sin x = -f_3(x).
\end{aligned}$$

Since all four images are in V, it follows that the entire range of D is contained in V. Also, we can write down the matrix of D (regarded as an operator on V) relative to the basis f_1, f_2, f_3, f_4:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

To prove that the range $\mathcal{R}(D)$ coincides with V, it is enough to show that each of the functions f_1, f_2, f_3, f_4 is in $\mathcal{R}(D)$. Indeed,

$$\begin{aligned}
D(-f_2 + f_3) &= -D(f_2) + D(f_3) = -(-f_1 + f_4) + f_4 = f_1, \\
D(f_1 + f_4) &= D(f_1) + D(f_4) = (f_2 + f_3) + (-f_3) = f_2, \\
D(-f_4) &= -D(f_4) = -(-f_3) = f_3, \\
D(f_3) &= f_4.
\end{aligned}$$

By the Dimension Theorem, $\dim \mathcal{R}(D) + \dim \mathcal{N}(D) = \dim V$. Since the range of D is V, it follows that $\dim \mathcal{N}(D) = 0$. Thus the null-space $\mathcal{N}(D)$ is trivial.

Problem 4′ (25 pts.) The functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$ span a 4-dimensional subspace V of the vector space $\mathcal{F}(\mathbb{R})$. Consider a linear transformation $L : V \to \mathcal{F}(\mathbb{R})$ given by $(Lf)(x) = f(x+1)$, $x \in \mathbb{R}$ for all functions $f \in V$.

(i) Show that the range of L is V and the null-space of L is trivial.

(ii) Find the matrix of L (regarded as an operator on V) relative to the basis f_1, f_2, f_3, f_4.

Solution: the matrix of L is $\begin{pmatrix} \cos 1 & -\sin 1 & 0 & 0 \\ \sin 1 & \cos 1 & 0 & 0 \\ \cos 1 & -\sin 1 & \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 & \sin 1 & \cos 1 \end{pmatrix}$.

Since it is given that the functions f_1, f_2, f_3, f_4 span a 4-dimensional subspace, they must be linearly independent and form a basis for the subspace. First we compute the images of these functions under the transformation L:

$$\begin{aligned}
(Lf_1)(x) &= f_1(x+1) = (x+1)\sin(x+1) = (x+1)(\sin x\,\cos 1 + \cos x\,\sin 1)\\
&= (\cos 1)f_1(x) + (\sin 1)f_2(x) + (\cos 1)f_3(x) + (\sin 1)f_4(x),\\
(Lf_2)(x) &= f_2(x+1) = (x+1)\cos(x+1) = (x+1)(\cos x\,\cos 1 - \sin x\,\sin 1)\\
&= (-\sin 1)f_1(x) + (\cos 1)f_2(x) + (-\sin 1)f_3(x) + (\cos 1)f_4(x),\\
(Lf_3)(x) &= f_3(x+1) = \sin(x+1) = \sin x\,\cos 1 + \cos x\,\sin 1\\
&= (\cos 1)f_3(x) + (\sin 1)f_4(x),\\
(Lf_4)(x) &= f_4(x+1) = \cos(x+1) = \cos x\,\cos 1 - \sin x\,\sin 1\\
&= (-\sin 1)f_3(x) + (\cos 1)f_4(x).
\end{aligned}$$

Since all four images are in V, it follows that the entire range of L is contained in V. Also, we can write down the matrix of L (regarded as an operator on V) relative to the basis f_1, f_2, f_3, f_4:

$$\begin{pmatrix} \cos 1 & -\sin 1 & 0 & 0\\ \sin 1 & \cos 1 & 0 & 0\\ \cos 1 & -\sin 1 & \cos 1 & -\sin 1\\ \sin 1 & \cos 1 & \sin 1 & \cos 1 \end{pmatrix}.$$

It follows from the definition of the operator L that the function Lf is identically zero only if f is identically zero. Hence the null-space of L is trivial.

By the Dimension Theorem, $\dim \mathcal{R}(L) + \dim \mathcal{N}(L) = \dim V$. Since the null-space of L is trivial, we have $\dim \mathcal{N}(L) = 0$ so that $\dim \mathcal{R}(L) = \dim V$. Since the range $\mathcal{R}(L)$ is contained in V, it follows that $\mathcal{R}(L) = V$.

Bonus Problem 5 (15 pts.) The set $\mathbb{R}_+$ of positive real numbers is a (real) vector space with respect to unusual operations of addition and scalar multiplication given by $x \oplus y = xy$ and $r \odot x = x^r$ for all $x, y \in \mathbb{R}_+$ and $r \in \mathbb{R}$. Prove that this vector space is isomorphic to $\mathbb{R}$ (with usual linear operations).

An isomorphism is provided by the logarithmic function $f(x) = \log x$ (to any base). Indeed, f is a one-to-one mapping of $\mathbb{R}_+$ onto $\mathbb{R}$. Since $\log(xy) = \log x + \log y$ for any $x, y > 0$, we have $f(x \oplus y) = f(x) + f(y)$. Since $\log x^r = r \log x$ for any $x > 0$ and $r \in \mathbb{R}$, we have $f(r \odot x) = rf(x)$. Thus f is a linear mapping.

Bonus Problem 5′ (15 pts.) Prove that the real numbers $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ are linearly independent over $\mathbb{Q}$.

Assume that $a\sqrt{2} + b\sqrt{3} + c\sqrt{6} = 0$ for some rational numbers a, b, and c. We have to prove that $a = b = c = 0$.

Indeed, the equality $a\sqrt{2} + b\sqrt{3} + c\sqrt{6} = 0$ is equivalent to $a\sqrt{2} + b\sqrt{3} = -c\sqrt{6}$. Squaring both sides of the latter, we obtain $(a\sqrt{2} + b\sqrt{3})^2 = (-c\sqrt{6})^2$. After simplification, $2ab\sqrt{6} + 2a^2 + 3b^2 = 6c^2$. Since the numbers $2ab$, $2a^2 + 3b^2$, and $6c^2$ are rational while $\sqrt{6}$ is not, it follows that $2ab = 0$. Then $a = 0$ or $b = 0$. In the first case, we have $b\sqrt{3} + c\sqrt{6} = 0$, which implies that $b = 0$ as otherwise $1/\sqrt{2} = -c/b$, a rational number. In the second case, we have $a\sqrt{2} + c\sqrt{6} = 0$, which implies that $a = 0$ as otherwise $1/\sqrt{3} = -c/a$, a rational number. Thus $a = b = 0$ in any case. Then $c = 0$ as well.