MATH 423–200/500 Spring 2012

Sample problems for Test 2: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) Find a cubic polynomial $p(x)$ such that $p(-2) = 0$, $p(-1) = 4$, $p(1) = 0$, and $p(2) = 4$.

Let $p(x) = a+bx+cx^2+dx^3$. Then $p(-2) = a-2b+4c-8d$, $p(-1) = a-b+c-d$, $p(1) = a+b+c+d$, and $p(2) = a + 2b + 4c + 8d$. The coefficients a, b, c, and d are to be chosen so that

$$\begin{cases} a - 2b + 4c - 8d = 0, \\ a - b + c - d = 4, \\ a + b + c + d = 0, \\ a + 2b + 4c + 8d = 4. \end{cases}$$

This is a system of linear equations. Let us convert its augmented matrix to reduced row echelon form using elementary row operations:

$$\left(\begin{array}{rrrr|r} 1 & -2 & 4 & -8 & 0 \\ 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 8 & 4 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 4 \\ 1 & -2 & 4 & -8 & 0 \\ 1 & 2 & 4 & 8 & 4 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 4 \\ 1 & -2 & 4 & -8 & 0 \\ 1 & 2 & 4 & 8 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 4 \\ 0 & -3 & 3 & -9 & 0 \\ 1 & 2 & 4 & 8 & 4 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 4 \\ 0 & -3 & 3 & -9 & 0 \\ 0 & 1 & 3 & 7 & 4 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & -3 & 3 & -9 & 0 \\ 0 & 1 & 3 & 7 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 1 & 3 & 7 & 4 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 1 & 3 & 7 & 4 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 3 & 6 & 6 \end{array}\right)$$

$$\to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 3 & 6 & 6 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

It follows that $a = 2$, $b = -3$, $c = 0$, and $d = 1$. Thus $p(x) = x^3 - 3x + 2$.

Alternative solution: Since -2 and 1 are roots of the cubic polynomial p, it has the form $p(x) = (x+2)(x-1)(ax+b)$. Then $p(-1) = 2a - 2b$ and $p(2) = 8a + 4b$. Therefore a and b are to be chosen so that

$$\begin{cases} 2a - 2b = 4, \\ 8a + 4b = 4 \end{cases} \iff \begin{cases} a - b = 2, \\ 2a + b = 1. \end{cases}$$

Solving this system of linear equations, we obtain $a = 1$, $b = -1$. Thus

$$p(x) = (x+2)(x-1)(x-1) = (x+2)(x^2 - 2x + 1) = x^3 - 3x + 2.$$

Problem 2 (25 pts.) Evaluate a determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 \end{vmatrix}.$$

For which values of parameters c_1, c_2, c_3, c_4 is this determinant equal to zero?

Let d denote the value of the determinant. To simplify the matrix, we subtract c_1 times the 3rd row from the 4th row, then subtract c_1 times the 2nd row from the 3rd row, then subtract c_1 times the 1st row from the 2nd row:

$$d = \begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ 0 & c_2^3 - c_1c_2^2 & c_3^3 - c_1c_3^2 & c_4^3 - c_1c_4^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ 0 & c_2^2 - c_1c_2 & c_3^2 - c_1c_3 & c_4^2 - c_1c_4 \\ 0 & c_2^3 - c_1c_2^2 & c_3^3 - c_1c_3^2 & c_4^3 - c_1c_4^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & c_2 - c_1 & c_3 - c_1 & c_4 - c_1 \\ 0 & c_2^2 - c_1c_2 & c_3^2 - c_1c_3 & c_4^2 - c_1c_4 \\ 0 & c_2^3 - c_1c_2^2 & c_3^3 - c_1c_3^2 & c_4^3 - c_1c_4^2 \end{vmatrix}.$$

The expansion by the first column yields

$$d = \begin{vmatrix} c_2 - c_1 & c_3 - c_1 & c_4 - c_1 \\ c_2^2 - c_1c_2 & c_3^2 - c_1c_3 & c_4^2 - c_1c_4 \\ c_2^3 - c_1c_2^2 & c_3^3 - c_1c_3^2 & c_4^3 - c_1c_4^2 \end{vmatrix}.$$

Now there is a common factor in each column:

$$d = \begin{vmatrix} c_2 - c_1 & c_3 - c_1 & c_4 - c_1 \\ (c_2 - c_1)c_2 & (c_3 - c_1)c_3 & (c_4 - c_1)c_4 \\ (c_2 - c_1)c_2^2 & (c_3 - c_1)c_3^2 & (c_4 - c_1)c_4^2 \end{vmatrix} = (c_2 - c_1) \begin{vmatrix} 1 & c_3 - c_1 & c_4 - c_1 \\ c_2 & (c_3 - c_1)c_3 & (c_4 - c_1)c_4 \\ c_2^2 & (c_3 - c_1)c_3^2 & (c_4 - c_1)c_4^2 \end{vmatrix}$$

$$= (c_2 - c_1)(c_3 - c_1) \begin{vmatrix} 1 & 1 & c_4 - c_1 \\ c_2 & c_3 & (c_4 - c_1)c_4 \\ c_2^2 & c_3^2 & (c_4 - c_1)c_4^2 \end{vmatrix} = (c_2 - c_1)(c_3 - c_1)(c_4 - c_1) \begin{vmatrix} 1 & 1 & 1 \\ c_2 & c_3 & c_4 \\ c_2^2 & c_3^2 & c_4^2 \end{vmatrix}.$$

The latter determinant is evaluated using the same technique as before:

$$\begin{vmatrix} 1 & 1 & 1 \\ c_2 & c_3 & c_4 \\ c_2^2 & c_3^2 & c_4^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ c_2 & c_3 & c_4 \\ 0 & c_3^2 - c_2c_3 & c_4^2 - c_2c_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & c_3 - c_2 & c_4 - c_2 \\ 0 & c_3^2 - c_2c_3 & c_4^2 - c_2c_4 \end{vmatrix}$$

$$= \begin{vmatrix} c_3 - c_2 & c_4 - c_2 \\ c_3^2 - c_2c_3 & c_4^2 - c_2c_4 \end{vmatrix} = \begin{vmatrix} c_3 - c_2 & c_4 - c_2 \\ (c_3 - c_2)c_3 & (c_4 - c_2)c_4 \end{vmatrix} = (c_3 - c_2) \begin{vmatrix} 1 & c_4 - c_2 \\ c_3 & (c_4 - c_2)c_4 \end{vmatrix}$$

$$= (c_3 - c_2)(c_4 - c_2) \begin{vmatrix} 1 & 1 \\ c_3 & c_4 \end{vmatrix} = (c_3 - c_2)(c_4 - c_2)(c_4 - c_3).$$

Thus

$$d = (c_2 - c_1)(c_3 - c_1)(c_4 - c_1)(c_3 - c_2)(c_4 - c_2)(c_4 - c_3).$$

The determinant is equal to zero if and only if the numbers c_1, c_2, c_3, c_4 are not all distinct.

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.

The eigenvalues of A are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 2(1-\lambda) - 2(1-\lambda)$$

$$= (1-\lambda)\big((1-\lambda)^2 - 4\big) = (1-\lambda)\big((1-\lambda) - 2\big)\big((1-\lambda) + 2\big) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix A has three eigenvalues: -1, 1, and 3.

(ii) For each eigenvalue of A, find an associated eigenvector.

An eigenvector $\mathbf{v} = (x, y, z)^t$ of A associated with an eigenvalue λ is a nonzero solution of the vector equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$. To solve the equation, we apply row reduction to the matrix $A - \lambda I$.

First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = s$, $y = -s$, $z = s$, where $s \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1, -1, 1)^t$ is an eigenvector of A associated with the eigenvalue -1.

Secondly, consider the case $\lambda = 1$. The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -s$, $y = 0$, $z = s$, where $s \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, 0, 1)^t$ is an eigenvector of A associated with the eigenvalue 1.

Finally, consider the case $\lambda = 3$. The row reduction yields

$$A - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - 3I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = s$, $y = s$, $z = s$, where $s \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)^t$ is an eigenvector of A associated with the eigenvalue 3.

(iii) Find all eigenvalues of the matrix A^3.

Suppose that $\mathbf{v}$ is an eigenvector of the matrix A associated with an eigenvalue λ, that is, $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v},$$
$$A^3\mathbf{v} = A(A^2\mathbf{v}) = A(\lambda^2\mathbf{v}) = \lambda^2(A\mathbf{v}) = \lambda^2(\lambda\mathbf{v}) = \lambda^3\mathbf{v}.$$

Therefore $\mathbf{v}$ is also an eigenvector of the matrix A^3 and the associated eigenvalue is λ^3. We already know that the matrix A has eigenvalues -1, 1, and 3. It follows that A^3 has eigenvalues -1, 1, and 27. It remains to notice that a 3×3 matrix can have at most 3 eigenvalues.

Problem 4 (25 pts.) Let $B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. Find a matrix C such that $C^2 = B^2$, but $C \neq \pm B$.

First we diagonalize the matrix B. The characteristic polynomial is

$$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5).$$

It has roots 1 and 5.

An eigenvector $\mathbf{v} = (x, y)^t$ of B associated with the eigenvalue 1 satisfies

$$(B - I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + 3y = 0.$$

In particular, $\mathbf{v}_1 = (-3, 1)^t$ is one of the eigenvectors.

An eigenvector $\mathbf{v} = (x, y)^t$ of B associated with the eigenvalue 5 satisfies

$$(B - 5I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x - y = 0.$$

In particular, $\mathbf{v}_2 = (1, 1)^t$ is one of the eigenvectors.

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$. It follows that $B = UDU^{-1}$, where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad U = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now we let $C = UPU^{-1}$, where

$$P = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}.$$

The matrix P is chosen so that $P^2 = D^2$ and $P \neq \pm D$. Since $C^2 = UPU^{-1}UPU^{-1} = UP^2U^{-1}$ and $B^2 = UDU^{-1}UDU^{-1} = UD^2U^{-1}$, we obtain that $C^2 = B^2$ and $C \neq \pm B$.

It remains to compute the matrix C:

$$C = UPU^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 3 & 5 \\ -1 & 5 \end{pmatrix}\frac{1}{-4}\begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 5 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 18 \\ 6 & 14 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 9 \\ 3 & 7 \end{pmatrix}.$$

Bonus Problem 5 (15 pts.) Let X be a square matrix that can be represented as a block matrix

$$X = \begin{pmatrix} A & C \\ O & B \end{pmatrix},$$

where A and B are square matrices and O is a zero matrix. Prove that $\det(X) = \det(A)\det(B)$.

Consider block matrices

$$Y = \begin{pmatrix} I & C \\ O & B \end{pmatrix}, \qquad Z = \begin{pmatrix} A & O' \\ O & I' \end{pmatrix},$$

where I and I' are the identity matrices of the same dimensions as A and B, respectively, and O' is the zero matrix of the same dimensions as C. Multiplying Y and Z as block matrices, we obtain

$$YZ = \begin{pmatrix} IA + CO & IO' + CI' \\ OA + BO & OO' + BI' \end{pmatrix} = \begin{pmatrix} A & C \\ O & B \end{pmatrix} = X.$$

As a consequence, $\det(X) = \det(Y)\det(Z)$. It remains to show that $\det(Y) = \det(B)$ and $\det(Z) = \det(A)$. The determinant of the matrix Y is easily expanded by the first column:

$$\det(Y) = \left|\begin{array}{cccc|ccc} 1 & 0 & \dots & 0 & c_{11} & \dots & c_{1n} \\ 0 & 1 & \dots & 0 & c_{21} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & c_{k1} & \dots & c_{kn} \\ \hline 0 & 0 & \dots & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & & B & \\ 0 & 0 & \dots & 0 & & & \end{array}\right| = \left|\begin{array}{ccc|ccc} 1 & \dots & 0 & c_{21} & \dots & c_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & c_{k1} & \dots & c_{kn} \\ \hline 0 & \dots & 0 & & & \\ \vdots & \ddots & \vdots & & B & \\ 0 & \dots & 0 & & & \end{array}\right|.$$

The new determinant can also be expanded by the first column. We keep expanding and eventually obtain that $\det(Y) = \det(B)$. Similarly, the equality $\det(Z) = \det(A)$ is established by repeatedly expanding the determinant of Z along the last row.