MATH 423–500/200 March 30, 2012

Test 2: Solutions

Problem 1 (20 pts.) Find the determinant of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Solution: $\det A = 4$.

Let us modify the first row of A adding to it all other rows. These elementary row operations do not change the determinant:

$$\det A = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 4 & 4 & 4 & 4 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

Now all entries in the first row are the same:

$$\begin{vmatrix} 4 & 4 & 4 & 4 & 4 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}.$$

Finally, we subtract the first row of the latter matrix from every other row. These elementary row operations, which do not change the determinant, result in an upper triangular matrix:

$$\det A = 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix} = 4.$$

Problem 1′ (20 pts.) Find the determinant of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Solution: $\det A = -5$.

Problem 2 (25 pts.) Consider a system of linear equations in variables x, y, z:

$$\begin{cases} x + 2y - z = 1, \\ 2x + 3y + z = 3, \\ x + 3y + az = 0, \\ x + y + 2z = b. \end{cases}$$

Find values of parameters a and b for which the system has infinitely many solutions, and solve the system for these values.

Solution: $a = -4$, $b = 2$. General solution of the system for these values of parameters: $(x, y, z) = (3, -1, 0) + t(-5, 3, 1)$, $t \in \mathbb{R}$.

To determine the number of solutions for the system, we convert its augmented matrix to row echelon form using elementary row operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & a & 0 \\ 1 & 1 & 2 & b \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 1 & 3 & a & 0 \\ 1 & 1 & 2 & b \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & a+1 & -1 \\ 1 & 1 & 2 & b \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & a+1 & -1 \\ 0 & -1 & 3 & b-1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & a+4 & 0 \\ 0 & -1 & 3 & b-1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & a+4 & 0 \\ 0 & 0 & 0 & b-2 \end{array}\right).$$

Now the augmented matrix is in row echelon form (except for the case $a = -4$, $b \neq 2$ when one also needs to exchange the last two rows). If $b \neq 2$, then there is a leading entry in the rightmost column, which indicates inconsistency. In the case $b = 2$ the system is consistent. If, additionally, $a \neq -4$ then there is a leading entry in each of the first three columns, which implies uniqueness of the solution.

Thus the system has infinitely many solutions only if $a = -4$ and $b = 2$. To find the solutions, we proceed to reduced row echelon form (for these particular values of parameters):

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The latter matrix is the augmented matrix of the following system of linear equations equivalent to the given one:

$$\begin{cases} x + 5z = 3, \\ y - 3z = -1 \end{cases} \iff \begin{cases} x = -5z + 3, \\ y = 3z - 1. \end{cases}$$

The general solution is $(x, y, z) = (-5t + 3, 3t - 1, t) = (3, -1, 0) + t(-5, 3, 1)$, $t \in \mathbb{R}$.

Problem 3 (20 pts.) Let $B = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. Find a polynomial $p(x)$ such that $B^{-1} = p(B)$.

Solution: $p(x) = \dfrac{1}{5}x - \dfrac{4}{5}$.

By the Cayley-Hamilton theorem, $q(B) = O$, where q is the characteristic polynomial of the matrix B. We have

$$q(\lambda) = \det(B - \lambda I) = \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 3^2 = \lambda^2 - 4\lambda - 5.$$

Therefore $B^2 - 4B - 5I = O$. Then $(B - 4I)B = B^2 - 4B = 5I$ so that $\frac{1}{5}(B - 4I)B = I$. It follows that $B^{-1} = \frac{1}{5}(B - 4I) = \frac{1}{5}B - \frac{4}{5}I$.

Problem 3′ (20 pts.) Let $B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Find a polynomial $p(x)$ such that $B^{-1} = p(B)$.

Solution: $p(x) = -\dfrac{1}{5}x + \dfrac{6}{5}$.

Problem 4 (25 pts.) Let $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix C.
(ii) For each eigenvalue of C, find an associated eigenvector.
(iii) Find a diagonal matrix D and an invertible matrix U such that $C = UDU^{-1}$.

Solution: Eigenvalues of C: 0, 2, and 3. Associated eigenvectors: $(-1, 0, 1)$, $(1, 0, 1)$, and $(0, 1, 0)$, respectively.

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Problem 4′ (25 pts.) Let $C = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix C.
(ii) For each eigenvalue of C, find an associated eigenvector.
(iii) Find a diagonal matrix D and an invertible matrix U such that $C = UDU^{-1}$.

Solution: Eigenvalues of C: -2, 0, and 2. Associated eigenvectors: $(-1, 0, 1)$, $(1, 0, 1)$, and $(0, 1, 0)$, respectively.

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Bonus Problem 5 (15 pts.) Let A be the matrix from Problem 1.

(i) Find all eigenvalues of A.
(ii) For each eigenvalue of A, find a basis for the associated eigenspace.

Solution: Eigenvalues of A: 4 and -1. Basis for the eigenspace associated with the eigenvalue 4: $\{(1,1,1,1,1)\}$. Basis for the eigenspace associated with the eigenvalue -1: $\{(-1,1,0,0,0),(-1,0,1,0,0),(-1,0,0,1,0),(-1,0,0,0,1)\}$.

The characteristic polynomial of the matrix A is computed in the same way as the determinant of A was evaluated in the solution of Problem 1 above:

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 & 1 \\ 1 & 1 & -\lambda & 1 & 1 \\ 1 & 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & 4-\lambda & 4-\lambda & 4-\lambda & 4-\lambda \\ 1 & -\lambda & 1 & 1 & 1 \\ 1 & 1 & -\lambda & 1 & 1 \\ 1 & 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & 1 & -\lambda \end{vmatrix}$$

$$= (4-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 & 1 \\ 1 & 1 & -\lambda & 1 & 1 \\ 1 & 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & 1 & -\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -\lambda-1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda-1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda-1 & 0 \\ 0 & 0 & 0 & 0 & -\lambda-1 \end{vmatrix}$$

$$= (4-\lambda)(-\lambda-1)^4 = (4-\lambda)(1+\lambda)^4.$$

The roots of this polynomial are 4 and -1. Since 4 is a simple root, the associated eigenspace is one-dimensional. It is easy to observe that $(1,1,1,1,1)$ is an eigenvector for 4. Therefore this vector forms a basis for the eigenspace.

All entries of the matrix $A+I$ are equal to 1. It follows that the eigenspace associated to the eigenvalue -1 consists of all vectors $(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5$ such that $x_1+x_2+x_3+x_4+x_5=0$. The general solution of this linear equation is

$$(x_1,x_2,x_3,x_4,x_5) = (-t_1-t_2-t_3-t_4, t_1, t_2, t_3, t_4)$$

$$= t_1(-1,1,0,0,0) + t_2(-1,0,1,0,0) + t_3(-1,0,0,1,0) + t_4(-1,0,0,0,1), \quad t_1,t_2,t_3,t_4 \in \mathbb{R}.$$

We obtain that vectors $(-1,1,0,0,0)$, $(-1,0,1,0,0)$, $(-1,0,0,1,0)$, and $(-1,0,0,0,1)$ form a basis for the eigenspace of A associated to the eigenvalue -1.

Bonus Problem 5′ (15 pts.) Let A be the matrix from Problem 1′.

(i) Find all eigenvalues of A.
(ii) For each eigenvalue of A, find a basis for the associated eigenspace.

Solution: Eigenvalues of A: 5 and -1. Basis for the eigenspace associated with the eigenvalue 5: $\{(1,1,1,1,1,1)\}$. Basis for the eigenspace associated with the eigenvalue -1: $\{(-1,1,0,0,0,0),(-1,0,1,0,0,0),(-1,0,0,1,0,0),(-1,0,0,0,1,0),(-1,0,0,0,0,1)\}$.