MATH 433

Applied Algebra

Lecture 3:
Mathematical induction.

Mathematical induction

Well-ordering principle: any nonempty set of positive integers has the smallest element. (Equivalently, any decreasing sequence of positive integers is finite.)

Induction principle: Let $P(n)$ be an assertion depending on the positive integer variable n. Suppose that

- $P(1)$ holds,
- whenever $P(k)$ holds, so does $P(k+1)$.

Then $P(n)$ holds for all positive integers n.

Remarks. The assertion $P(1)$ is called the **basis of induction**. The implication $P(k) \implies P(k+1)$ is called the **induction step**.

Examples of assertions $P(n)$:

(a) $1+2+\cdots+n = n(n+1)/2$,
(b) $n(n+1)(n+2)$ is divisible by 6,
(c) $n = 2p + 3q$ for some $p, q \in \mathbb{Z}$.

Theorem The well-ordering principle implies the induction principle.

Proof: Let $P(n)$ be an assertion depending on the positive integer variable n such that $P(1)$ holds and $P(k)$ implies $P(k+1)$ for any integer $k > 0$.

Consider the set $S = \{n \in \mathbb{P} : P(n) \text{ does not hold}\}$. Assume that S is not empty. By the well-ordering principle, the set S has the smallest element m. Since $P(1)$ holds, $m \neq 1$ so that $m - 1 > 0$. Clearly, $m - 1 \notin S$, therefore $P(m-1)$ holds. But $P(m-1) \implies P(m)$ so that $P(m)$ holds as well.

The contradiction means that the assumption was wrong. Thus the set S is empty.

Theorem $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Proof: Let us use the induction principle (on the variable n).

Basis of induction: check the formula for $n = 1$.
In this case, $1 = 1(1+1)/2$, which is true.

Induction step: assume that the formula is true for $n = m$ and derive it for $n = m + 1$.
Inductive assumption: $1 + 2 + \cdots + m = m(m+1)/2$.
Then

$$1 + 2 + \cdots + m + (m+1) = \frac{m(m+1)}{2} + (m+1)$$

$$= (m+1)\Big(\frac{m}{2} + 1\Big) = \frac{(m+1)(m+2)}{2}.$$

By the principle of mathematical induction, the formula holds for all $n \in \mathbb{P}$.

Strong induction principle: Let $P(n)$ be an assertion depending on a positive integer variable n. Suppose that $P(n)$ holds whenever $P(k)$ holds for all $k < n$. Then $P(n)$ holds for all positive integers n.

For $n = 1$, this means that $P(1)$ holds unconditionally.

For $n = 2$, this means that $P(1)$ implies $P(2)$.

For $n = 3$, this means that $P(1)$ and $P(2)$ imply $P(3)$.

And so on...

Strong induction

Theorem Let $P(n)$ be an assertion depending on a positive integer variable n. Suppose that $P(n)$ holds whenever $P(k)$ holds for all $k < n$. Then $P(n)$ holds for all $n \in \mathbb{P}$.

Proof of the theorem: For any $n \in \mathbb{P}$ we formulate new assertion $Q(n) =$ "$P(k)$ holds for any positive integer $k \leq n$". We are going to prove $Q(n)$ by (usual) induction on n.

First of all, $Q(1)$ holds since it is equivalent to $P(1)$. Now assume that $Q(n)$ holds for some $n \in \mathbb{P}$. By hypothesis of the theorem, $Q(n)$ implies $P(n+1)$. Moreover, $Q(n+1)$ holds if and only if both $Q(n)$ and $P(n+1)$ hold. Therefore $Q(n)$ implies $Q(n+1)$ for all $n \in \mathbb{P}$. By the principle of mathematical induction, $Q(n)$ holds for all $n \in \mathbb{P}$.

It remains to notice that $Q(n)$ implies $P(n)$ for all $n \in \mathbb{P}$.

Well-ordering and induction

Principle of well-ordering:
The set $\mathbb{P}$ is well-ordered, that is, any nonempty subset of $\mathbb{P}$ has a least element.

Principle of mathematical induction:
Let $P(n)$ be an assertion depending on a variable $n \in \mathbb{P}$. Suppose that $P(1)$ holds and $P(k)$ implies $P(k+1)$ for any $k \in \mathbb{P}$. Then $P(n)$ holds for all $n \in \mathbb{P}$.

Induction with a different base:
Let $P(n)$ be an assertion depending on an integer variable n. Suppose that $P(n_0)$ holds for some $n_0 \in \mathbb{Z}$ and $P(k)$ implies $P(k+1)$ for any $k \geq n_0$. Then $P(n)$ holds for all $n \geq n_0$.

Strong induction: Let $P(n)$ be an assertion depending on a variable $n \in \mathbb{P}$. Suppose that $P(n)$ holds whenever $P(k)$ holds for all $k < n$. Then $P(n)$ holds for all $n \in \mathbb{P}$.

Inductive definition

The principle of mathematical induction allows to define mathematical objects inductively (that is, recursively).

Examples of inductive definitions:

- Power a^n of a number

Given a real number a, we let $a^0 = 1$ and $a^n = a^{n-1}a$ for any $n \in \mathbb{P}$.

- Factorial $n!$

We let $0! = 1$ and $n! = (n-1)! \cdot n$ for any $n \in \mathbb{P}$.

- Fibonacci numbers $F_1, F_2, \ldots$

We let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for any $n \geq 3$.

Problem. Let $\{F_n\}$ be the Fibonacci numbers: $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for any $n \geq 3$. Prove that $(1.5)^{n-2} \leq F_n \leq 2^{n-1}$ for all $n \geq 1$.

Let us use the strong induction on n. In the case $n = 1$, we check the inequalities directly: $(1.5)^{1-2} \leq F_1 = 1 \leq 2^{1-1}$.
In the case $n = 2$, we also check them directly:
$(1.5)^{2-2} \leq F_2 = 1 \leq 2^{2-1}$.

Now consider an integer $m \geq 3$ and assume that the inequalities hold for all $n < m$. In particular, they hold for $n = m - 1$ and $n = m - 2$. Then

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} \leq 2^{(m-1)-1} + 2^{(m-2)-1} = 2^{m-2} + 2^{m-3} \\ &= 2^{m-1}(1/2 + 1/4) < 2^{m-1}, \end{aligned}$$

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} \geq (1.5)^{(m-1)-2} + (1.5)^{(m-2)-2} \\ &= (1.5)^{m-3} + (1.5)^{m-4} = (1.5)^{m-2}(2/3 + 4/9) > (1.5)^{m-2}. \end{aligned}$$

The induction is complete.