MATH 433

Applied Algebra

Lecture 14: Functions. Relations.

Cartesian product

Definition. The **Cartesian product** $X \times Y$ of two sets X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

The Cartesian square $X \times X$ is also denoted X^2.

If the sets X and Y are finite, then $\#(X \times Y) = (\#X)(\#Y)$, where $\#S$ denote the number of elements in a set S.

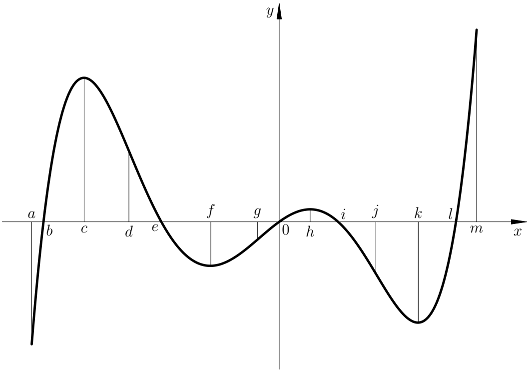
Functions

A **function** $f : X \to Y$ is an assignment: to each $x \in X$ we assign an element $f(x) \in Y$.

The **graph** of the function $f : X \to Y$ is defined as the subset of $X \times Y$ consisting of all pairs of the form $(x, f(x))$, $x \in X$.

Definition. A function $f : X \to Y$ is **surjective** (or **onto**) if for each $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$.
The function f is **injective** (or **one-to-one**) if $f(x') = f(x) \implies x' = x$.
Finally, f is **bijective** if it is both surjective and injective. Equivalently, if for each $y \in Y$ there is exactly one $x \in X$ such that $f(x) = y$.

The inverse function f^{-1} exists if and only if f is bijective.



y
x
a
b
c
d
e
f
g
0
h
i
j
k
l
m

Relations

Definition. Let X and Y be sets. A **relation** R from X to Y is given by specifying a subset of the Cartesian product: $S_R \subset X \times Y$.

If $(x, y) \in S_R$, then we say that x **is related to** y (in the sense of R or by R) and write xRy.

Remarks. • Usually the relation R is identified with the set S_R.
• In the case $X = Y$, the relation R is called a **relation on** X.

Examples. • "is equal to"
$xRy \iff x = y$

• "is not equal to"
$xRy \iff x \neq y$

• "is mapped by f to"
$xRy \iff y = f(x)$, where $f : X \to Y$ is a function.
Equivalently, R is the graph of the function f.

• "is the image under f of"
(from Y to X) $yRx \iff y = f(x)$, where $f : X \to Y$ is a function. If f is invertible, then R is the graph of f^{-1}.

• reversed R'
$xRy \iff yR'x$, where R' is a relation from Y to X.

• not R'
$xRy \iff$ not $xR'y$, where R' is a relation from X to Y.
Equivalently, $R = (X \times Y) \setminus R'$ (set difference).

Relations on a set

- "is equal to"

$xRy \iff x = y$

- "is not equal to"

$xRy \iff x \neq y$

- "is less than"

$X = \mathbb{R},\ \ xRy \iff x < y$

- "is less than or equal to"

$X = \mathbb{R},\ \ xRy \iff x \leq y$

- "is contained in"

$X =$ the set of all subsets of some set Y,
$xRy \iff x \subset y$

- "is congruent modulo n to"

$X = \mathbb{Z},\ \ xRy \iff x \equiv y \mod n$

- "divides"

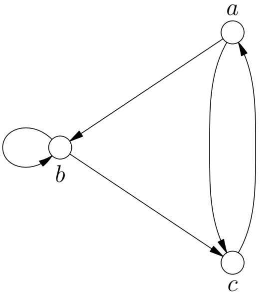
$X = \mathbb{P},\ \ xRy \iff x|y$

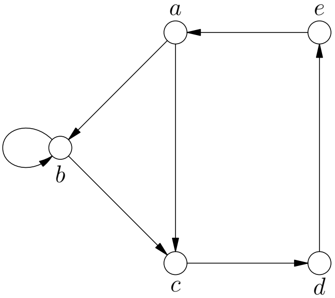
A relation R on a finite set X can be represented by a **directed graph**.

Vertices of the graph are elements of X, and we have a directed edge from x to y if and only if xRy.

Another way to represent the relation R is the **adjacency table**.

Rows and columns are labeled by elements of X. We put 1 at the intersection of a row x with a column y if xRy. Otherwise we put 0.





	a	b	c
a	0	1	1
b	0	1	1
c	1	0	0

	a	b	c	d	e
a	0	1	1	0	0
b	0	1	1	0	0
c	0	0	0	1	0
d	0	0	0	0	1
e	1	0	0	0	0

Properties of relations

Definition. Let R be a relation on a set X. We say that R is

- **reflexive** if xRx for all $x \in X$,
- **symmetric** if, for all $x, y \in X$, xRy implies yRx,
- **antisymmetric** if, for all $x, y \in X$, xRy and yRx cannot hold simultaneously,
- **weakly antisymmetric** if, for all $x, y \in X$, xRy and yRx imply that $x = y$,
- **transitive** if, for all $x, y, z \in X$, xRy and yRz imply that xRz.

Partial ordering

Definition. A relation R on a set X is a **partial ordering** (or **partial order**) if R is reflexive, weakly antisymmetric, and transitive:

- xRx,
- xRy and $yRx \implies x = y$,
- xRy and $yRz \implies xRz$.

A relation R on a set X is a **strict partial order** if R is antisymmetric and transitive:

- $xRy \implies$ not yRx,
- xRy and $yRz \implies xRz$.

Examples. "is less than or equal to", "is contained in", "is a divisor of" are partial orders. "is less than" is a strict order.

Equivalence relation

Definition. A relation R on a set X is an **equivalence relation** if R is reflexive, symmetric, and transitive:

- xRx,
- $xRy \implies yRx$,
- xRy and $yRz \implies xRz$.

Examples. "is equal to", "is congruent modulo n to" are equivalence relations.

Given an equivalence relation R on X, the **equivalence class** of an element $x \in X$ relative to R is the set of all elements $y \in X$ such that yRx.

Theorem The equivalence classes form a **partition** of the set X, which means that

- any two equivalence classes either coincide, or else they are disjoint,
- any element of X belongs to some equivalence class.