MATH 433

Applied Algebra

Lecture 21:
Transformation groups.

Abstract groups

Definition. A **group** is a set G, together with a binary operation $*$, that satisfies the following axioms:

(G1: closure)
for all elements g and h of G, $g * h$ is an element of G;

(G2: associativity)
$(g * h) * k = g * (h * k)$ for all $g, h, k \in G$;

(G3: existence of identity)
there exists an element $e \in G$, called the **identity** (or **unit**) of G, such that $e * g = g * e = g$ for all $g \in G$;

(G4: existence of inverse)
for every $g \in G$ there exists an element $h \in G$, called the **inverse** of g, such that $g * h = h * g = e$.

The group $(G, *)$ is said to be **commutative** (or **Abelian**) if it satisfies an additional axiom:

(G5: commutativity) $g * h = h * g$ for all $g, h \in G$.

Transformation groups

Definition. A **transformation group** is a group of bijective transformations of a set X with the operation of composition.

Examples.

- Symmetric group $S(n)$: all permutations of $\{1, 2, \dots, n\}$.
- Alternating group $A(n)$: even permutations of $\{1, 2, \dots, n\}$.
- $\mathrm{Homeo}(\mathbb{R})$: the group of all invertible functions $f : \mathbb{R} \to \mathbb{R}$ such that both f and f^{-1} are continuous (such functions are called **homeomorphisms**).
- $\mathrm{Homeo}^+(\mathbb{R})$: the group of all increasing functions in $\mathrm{Homeo}(\mathbb{R})$ (i.e., those that preserve orientation of the real line).
- $\mathrm{Diff}(\mathbb{R})$: the group of all invertible functions $f : \mathbb{R} \to \mathbb{R}$ such that both f and f^{-1} are continuously differentiable (such functions are called **diffeomorphisms**).

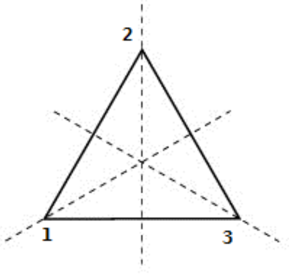
Groups of symmetries

Definition. A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a **motion** (or a **rigid motion**) if it preserves distances between points.

Theorem All motions of $\mathbb{R}^n$ form a transformation group. Any motion $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix.

Given a geometric figure $F \subset \mathbb{R}^n$, a **symmetry** of F is a motion of $\mathbb{R}^n$ that preserves F. All symmetries of F form a transformation group.

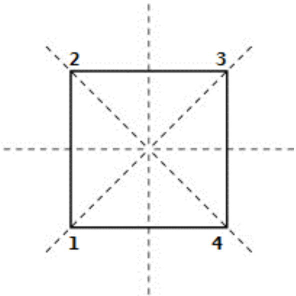
The **dihedral group** $D(n)$ is the group of symmetries of a regular n-gon. It consists of $2n$ elements: n reflections, $n-1$ rotations by angles $2\pi k/n$, $k = 1, 2, \ldots, n-1$, and the identity function.



Equlateral triangle

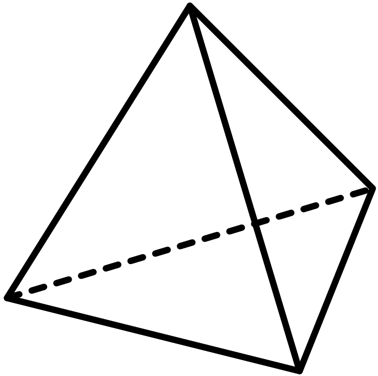
Any symmetry of a polygon maps vertices to vertices. Therefore it induces a permutation on the set of vertices. Moreover, the symmetry is uniquely recovered from the permutation.

In the case of the equilateral triangle, any permutation of vertices comes from a symmetry.



Square

In the case of the square, not every permutation of vertices comes from a symmetry of the square. The reason is that a symmetry must map adjacent vertices to adjacent vertices.



Regular tetrahedron

Any symmetry of a polyhedron maps vertices to vertices. In the case of the regular tetrahedron, any permutation of vertices comes from a symmetry.

Matrix groups

A group is called **linear** if its elements are $n \times n$ matrices and the group operation is matrix multiplication.

- **General linear group** $GL(n, \mathbb{R})$ consists of all $n \times n$ matrices that are invertible (i.e., with nonzero determinant).

  The identity element is $I = \mathrm{diag}(1, 1, \ldots, 1)$.

- **Special linear group** $SL(n, \mathbb{R})$ consists of all $n \times n$ matrices with determinant 1.

  Closed under multiplication since $\det(AB) = \det(A) \det(B)$.
  Also, $\det(A^{-1}) = (\det(A))^{-1}$.

- **Orthogonal group** $O(n, \mathbb{R})$ consists of all orthogonal $n \times n$ matrices $(A^T A = AA^T = I)$.

- **Special orthogonal group** $SO(n, \mathbb{R})$ consists of all orthogonal $n \times n$ matrices with determinant 1.

  $SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap SL(n, \mathbb{R})$.