MATH 433

Applied Algebra

Lecture 30:
Isomorphism of groups.
Classification of Abelian groups.

Quotient group

Consider an arbitrary group G (with multiplicative operation) and a subgroup H of G. Let G/H denote the set of all cosets gH of the subgroup H in G. We try to define an operation on G/H by the rule $(aH)(bH) = (ab)H$. Assume that this operation is well defined (it need not be). Then it makes G/H into a group, which is called the **quotient group** of G by the subgroup H.

Question. When the operation on G/H is well defined?

Suppose a' is another representative of the coset aH and b' is another representative of the coset bH. The operation on G/H is well defined if $a'b'$ is in the same coset as ab, or, equivalently, $(ab)^{-1}(a'b') \in H$. We have $a' = ah_1$ and $b' = bh_2$ for some $h_1, h_2 \in H$. Then $(ab)^{-1}(a'b') = b^{-1}a^{-1}ah_1bh_2 = (b^{-1}h_1b)h_2$.

Definition. The subgroup H of G is called **normal** if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

Homomorphism of groups

Definition. Let G and H be groups. A function $f : G \to H$ is called a **homomorphism** of the groups if $f(g_1 g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$.

Properties of homomorphisms:

- The identity element e_G in G is mapped to the identity element e_H in H.
- $f(g^{-1}) = (f(g))^{-1}$ for all $g \in G$.
- If K is a subgroup of G, then $f(K)$ is a subgroup of H.
- If L is a subgroup of H, then $f^{-1}(L)$ is a subgroup of G.
- If L is a normal subgroup of H, then $f^{-1}(L)$ is a normal subgroup of G.
- $f^{-1}(e_H)$ is a normal subgroup of G called the **kernel** of f.

Isomorphism of groups

Definition. Let G and H be groups. A function $f : G \to H$ is called an **isomorphism** of the groups if it is bijective and $f(g_1 g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$.

Theorem Isomorphism is an equivalence relation on the set of all groups.

Classification of groups consists of describing all equivalence classes of this relation and placing every known group into an appropriate class.

Theorem The following properties of groups are preserved under isomorphisms:

- the number of elements,
- being Abelian,
- being cyclic,
- having a subgroup of a particular order,
- having an element of a particular order.

Classification of finite Abelian groups

Given two groups G and H, the **direct product** $G \times H$ is the set of all ordered pairs (g, h), where $g \in G$, $h \in H$, with an operation defined by $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$.

The set $G \times H$ is a group under this operation. The identity element is (e_G, e_H), where e_G is the identity element in G and e_H is the identity element in H. The inverse of (g, h) is (g^{-1}, h^{-1}), where g^{-1} is computed in G and h^{-1} is computed in H.

Similarly, we can define the direct product $G_1 \times G_2 \times \cdots \times G_n$ of any finite collection of groups $G_1, G_2, \ldots, G_n$.

Theorem Any finite Abelian group is isomorphic to a direct product of cyclic groups $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$. Moreover, we can assume that the orders $n_1, n_2, \ldots, n_k$ of the cyclic groups are prime powers, in which case this direct product is unique (up to rearrangement of the factors).