MATH 433

Applied Algebra

Lecture 18:
Sign of a permutation.
Definition of the determinant.

Permutations

Let X be a finite set. A **permutation** of X is a bijection from X to itself. The set of all permutations of $\{1, 2, \ldots, n\}$ is called the **symmetric group** on n symbols and denoted $S(n)$.

Theorem Any permutation can be expressed as a product of disjoint cycles. This **cycle decomposition** is unique up to rearrangement of the cycles involved.

Theorem Let π be a permutation. Then there is a positive integer m such that $\pi^m = \mathrm{id}$.

The **order** of a permutation π, denoted $o(\pi)$, is defined as the smallest positive integer m such that $\pi^m = \mathrm{id}$.

Theorem Let $\pi \in S(n)$ and suppose that $\pi = \sigma_1\sigma_2\ldots\sigma_k$ is a decomposition of π as a product of disjoint cycles. Then the order of π is the least common multiple of the lengths of cycles $\sigma_1, \ldots, \sigma_k$.

Sign of a permutation

Theorem 1 (i) Any permutation is a product of transpositions.
(ii) If $\pi = \tau_1\tau_2\ldots\tau_n = \tau'_1\tau'_2\ldots\tau'_m$, where τ_i, τ'_j are transpositions, then the numbers n and m are of the same parity (that is, both even or both odd).

A permutation π is called **even** if it is a product of an even number of transpositions, and **odd** if it is a product of an odd number of transpositions.

The **sign** $\operatorname{sgn}(\pi)$ of the permutation π is defined to be $+1$ if π is even, and -1 if π is odd.

Theorem 2 (i) $\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$ for any $\pi, \sigma \in S(n)$.
(ii) $\operatorname{sgn}(\pi^{-1}) = \operatorname{sgn}(\pi)$ for any $\pi \in S(n)$.
(iii) $\operatorname{sgn}(\mathrm{id}) = 1$.
(iv) $\operatorname{sgn}(\tau) = -1$ for any transposition τ.
(v) $\operatorname{sgn}(\sigma) = (-1)^{r-1}$ for any cycle σ of length r.

Let $\pi \in S(n)$ and i, j be integers, $1 \leq i < j \leq n$. We say that the permutation π preserves order of the pair (i, j) if $\pi(i) < \pi(j)$. Otherwise π makes an **inversion**. Denote by $N(\pi)$ the number of inversions made by the permutation π.

Lemma 1 Let $\tau, \pi \in S(n)$ and suppose that τ is an adjacent transposition, $\tau = (k\ k{+}1)$. Then $|N(\tau\pi) - N(\pi)| = 1$.

Proof: For every pair (i, j), $1 \leq i < j \leq n$, let us compare the order of pairs $\pi(i), \pi(j)$ and $\tau\pi(i), \tau\pi(j)$. We observe that the order differs exactly for one pair, when $\{\pi(i), \pi(j)\} = \{k, k{+}1\}$. The lemma follows.

Lemma 2 Let $\pi \in S(n)$ and $\tau_1, \tau_2, \ldots, \tau_k$ be adjacent transpositions. Then **(i)** for any $\pi \in S(n)$ the numbers k and $N(\tau_1\tau_2 \ldots \tau_k\pi) - N(\pi)$ are of the same parity,
(ii) the numbers k and $N(\tau_1\tau_2 \ldots \tau_k)$ are of the same parity.

Sketch of the proof: **(i)** follows from Lemma 1 by induction on k. **(ii)** is a particular case of part (i), when $\pi = \mathrm{id}$.

Lemma 3 **(i)** Any cycle of length r is a product of $r-1$ transpositions. **(ii)** Any transposition is a product of an odd number of adjacent transpositions.

Proof: **(i)** $(x_1\ x_2\ \dots\ x_r) = (x_1\ x_2)(x_2\ x_3)(x_3\ x_4)\dots(x_{r-1}\ x_r)$.

(ii) $(k\ k+r) = \sigma^{-1}(k\ k+1)\sigma$, where $\sigma = (k+1\ k+2\ \dots\ k+r)$. By the above, $\sigma = (k+1\ k+2)(k+2\ k+3)\dots(k+r-1\ \ k+r)$ and $\sigma^{-1} = (k+r\ \ k+r-1)\dots(k+3\ k+2)(k+2\ k+1)$.

Theorem **(i)** Any permutation is a product of transpositions. **(ii)** If $\pi = \tau_1\tau_2\dots\tau_k$, where τ_i are transpositions, then the numbers k and $N(\pi)$ are of the same parity.

Proof: **(i)** Any permutation is a product of disjoint cycles. By Lemma 3, any cycle is a product of transpositions.

(ii) By Lemma 3, each of $\tau_1, \tau_2, \dots, \tau_k$ is a product of an odd number of adjacent transpositions. Hence $\pi = \tau_1'\tau_2'\dots\tau_m'$, where τ_i' are adjacent transpositions and number m is of the same parity as k. By Lemma 2, m has the same parity as $N(\pi)$.

Examples

- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 4 & 7 & 9 & 1 & 12 & 5 & 11 & 3 & 10 & 6 & 8 \end{pmatrix}$.

First we decompose π into a product of disjoint cycles:

$$\pi = (1\ 2\ 4\ 9\ 3\ 7\ 5)(6\ 12\ 8\ 11).$$

The cycle $\sigma_1 = (1\ 2\ 4\ 9\ 3\ 7\ 5)$ has length 7, hence it is an even permutation. The cycle $\sigma_2 = (6\ 12\ 8\ 11)$ has length 4, hence it is an odd permutation. Then

$$\mathrm{sgn}(\pi) = \mathrm{sgn}(\sigma_1\sigma_2) = \mathrm{sgn}(\sigma_1)\,\mathrm{sgn}(\sigma_2) = 1 \cdot (-1) = -1.$$

- $\pi = (2\ 4\ 3)(1\ 2)(2\ 3\ 4)$.

π is represented as a product of cycles. The transposition has sign -1 while the cycles of length 3 have sign $+1$. Even though the cycles are not disjoint, $\mathrm{sgn}(\pi) = 1 \cdot (-1) \cdot 1 = -1$.

Definition of the determinant

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

If $A = (a_{ij})$ is an $n \times n$ matrix then

$$\det A = \sum_{\pi \in S(n)} \operatorname{sgn}(\pi)\, a_{1,\pi(1)}\, a_{2,\pi(2)} \ldots a_{n,\pi(n)},$$

where π runs over all permutations of $\{1, 2, \ldots, n\}$.

Theorem $\det A^T = \det A$.

Proof: Let $A = (a_{ij})_{1\le i,j\le n}$. Then $A^T = (b_{ij})_{1\le i,j\le n}$, where $b_{ij} = a_{ji}$. We have

$$\det A^T = \sum_{\pi\in S(n)} \operatorname{sgn}(\pi)\, b_{1,\pi(1)}\, b_{2,\pi(2)} \dots b_{n,\pi(n)}$$
$$= \sum_{\pi\in S(n)} \operatorname{sgn}(\pi)\, a_{\pi(1),1}\, a_{\pi(2),2} \dots a_{\pi(n),n}$$
$$= \sum_{\pi\in S(n)} \operatorname{sgn}(\pi)\, a_{1,\pi^{-1}(1)}\, a_{2,\pi^{-1}(2)} \dots a_{n,\pi^{-1}(n)}.$$

When π runs over all permutations of $\{1,2,\dots,n\}$, so does $\sigma = \pi^{-1}$. It follows that

$$\det A^T = \sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma^{-1})\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \dots a_{n,\sigma(n)}$$
$$= \sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \dots a_{n,\sigma(n)} = \det A.$$

Theorem 1 Suppose A is a square matrix and B is obtained from A by exchanging two rows. Then $\det B = -\det A$.

Theorem 2 Suppose A is a square matrix and B is obtained from A by permuting its rows. Then $\det B = \det A$ if the permutation is even and $\det B = -\det A$ if the permutation is odd.

Proof: Let $A = (a_{ij})_{1 \le i,j \le n}$ be an $n \times n$ matrix. Suppose that a matrix B is obtained from A by permuting its rows according to a permutation $\sigma \in S(n)$. Then $B = (b_{ij})_{1 \le i,j \le n}$, where $b_{\sigma(i),j} = a_{ij}$. Equivalently, $b_{ij} = a_{\sigma^{-1}(i),j}$. We have

$$\det B = \sum_{\pi \in S(n)} \operatorname{sgn}(\pi)\, b_{1,\pi(1)}\, b_{2,\pi(2)} \dots b_{n,\pi(n)}$$

$$= \sum_{\pi \in S(n)} \operatorname{sgn}(\pi)\, a_{\sigma^{-1}(1),\pi(1)}\, a_{\sigma^{-1}(2),\pi(2)} \dots a_{\sigma^{-1}(n),\pi(n)}$$

$$= \sum_{\pi \in S(n)} \operatorname{sgn}(\pi)\, a_{1,\pi\sigma(1)}\, a_{2,\pi\sigma(2)} \dots a_{n,\pi\sigma(n)} = \operatorname{sgn}(\sigma) \det A.$$

The Vandermonde determinant

Definition. The **Vandermonde determinant** is the determinant of the following matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix},$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Equivalently, $V = (a_{ij})_{1 \le i,j \le n}$, where $a_{ij} = x_i^{j-1}$.

Theorem

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Corollary Consider a polynomial

$$p(x_1, x_2, \dots, x_n) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Then

$$p(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}) = \mathrm{sgn}(\pi)\, p(x_1, x_2, \dots, x_n)$$

for any permutation $\pi \in S(n)$.