MATH 433

Applied Algebra

Lecture 4:
More on greatest common divisor.
Prime numbers.
Unique factorisation theorem.

Greatest common divisor

Given positive integers $a_1, a_2, \dots, a_n$, the **greatest common divisor** $\gcd(a_1, a_2, \dots, a_n)$ is the largest positive integer that divides $a_1, a_2, \dots, a_n$.

Theorem (i) $\gcd(a_1, a_2, \dots, a_n)$ is the smallest positive integer represented as an integral linear combination of $a_1, a_2, \dots, a_n$.
(ii) $\gcd(a_1, a_2, \dots, a_n)$ is divisible by any other common divisor of $a_1, a_2, \dots, a_n$.

Remark. The theorem can be proved in the same way as in the case $n = 2$ (see Lecture 2). Another approach is by induction on n using the fact that $\gcd(a_1, a_2, \dots, a_n) = \gcd(a_1, \gcd(a_2, \dots, a_n))$.

Prime numbers

A positive integer p is **prime** if it has exactly two positive divisors, namely, 1 and p.

Examples. 2, 3, 5, 7, 29, 41, 101.

A positive integer n is **composite** if it a product of two strictly smaller positive integers.

Examples. $6 = 2 \cdot 3,\ 16 = 4 \cdot 4,\ 125 = 5 \cdot 25.$

Any positive integer is either prime or composite or 1. **Prime factorisation** of a positive integer $n \geq 2$ is a decomposition of n into a product of primes.

Examples. • $120 = 12 \cdot 10 = (2 \cdot 6) \cdot (2 \cdot 5)$
$= (2 \cdot (2 \cdot 3)) \cdot (2 \cdot 5) = 2^3 \cdot 3 \cdot 5.$
• $144 = 12^2 = (2^2 \cdot 3)^2 = 2^4 \cdot 3^2.$

Sieve of Eratosthenes

The **sieve of Eratosthenes** is a method to find all primes from 2 to n:

(1) Write down all integers from 2 to n.
(2) Select the smallest integer k that is not selected or crossed out yet.
(3) Cross out all multiples of k.
(4) If not all numbers are selected or crossed out, return to step (2).

The prime numbers are those selected in the process.

Unique factorisation theorem

Theorem Any positive integer $n \geq 2$ admits a prime factorisation. This factorisation is unique up to rearranging the factors.

Corollary There are infinitely many prime numbers.

Idea of the proof: Let $p_1, p_2, \ldots, p_k$ be the first k primes. Consider the number $N = p_1 p_2 \ldots p_k + 1$. By construction, this number is not divisible by $p_1, p_2, \ldots, p_k$. But it does have a prime divisor, due to the theorem.

Problem. Suppose m is a positive integer such that

$$m = 2^4 p_1 p_2 p_3,$$
$$m + 100 = 5 q_1 q_2 q_3,$$
$$m + 200 = 23 r_1 r_2 r_3 r_4,$$

where p_i, q_j, r_k are prime numbers and, moreover, $p_i \neq 2$, $q_j \neq 5$, $r_k \neq 23$. Find m.

The prime decomposition of 100 is $2^2 \cdot 5^2$. Since the numbers $m + 100$ and 100 are divisible by 5, so are their difference m and their sum $m + 200$.

The prime decomposition of 200 is $2^3 \cdot 5^2$. Since the number m is divisible by $2^4 = 16$, it follows that $m + 100$ is divisible by $2^2 = 4$ while $m + 200$ is divisible by $2^3 = 8$.

By the above the prime decomposition of $m + 200$ contains $2^3 \cdot 5 \cdot 23$. As there are only 5 factors in this decomposition, the number $m + 200$ is exactly $2^3 \cdot 5 \cdot 23 = 920$. Then $m + 100 = 820 = 2^2 \cdot 5 \cdot 41$ and $m = 720 = 2^4 \cdot 3^2 \cdot 5$.

Unique prime factorisation

Theorem Any positive integer $n \geq 2$ admits a prime factorisation. This factorisation is unique up to rearranging the factors.

Ideas of the proof: The **existence** is proved by strong induction on n. It is based on a simple fact: if $p_1 p_2 \dots p_s$ is a prime factorisation of k and $q_1 q_2 \dots q_t$ is a prime factorisation of m, then $p_1 p_2 \dots p_s q_1 q_2 \dots q_t$ is a prime factorisation of km.

The **uniqueness** is proved by (normal) induction on the number of prime factors. It is based on a (not so simple) fact: if a prime number p divides a product of primes $q_1 q_2 \dots q_t$ then one of the primes $q_1, \dots, q_t$ coincides with p.

Coprime numbers

Positive integers a and b are **relatively prime** (or **coprime**) if $\gcd(a, b) = 1$.

Theorem Suppose that a and b are coprime integers. Then
(i) $a|bc$ implies $a|c$;
(ii) $a|c$ and $b|c$ imply $ab|c$.

Idea of the proof: Since $\gcd(a, b) = 1$, there are integers m and n such that $ma + nb = 1$. Then $c = mac + nbc$.

Corollary 1 If a prime number p divides the product $b_1 b_2 \dots b_n$, then p divides one of the integers $b_1, \dots, b_n$.

Corollary 2 If an integer c is divisible by pairwise coprime integers $a_1, a_2, \dots, a_n$, then c is divisible by the product $a_1 a_2 \dots a_n$.

Let $a = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ and $b = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes and n_i, m_i are nonnegative integers.

Theorem (i) $ab = p_1^{n_1+m_1} p_2^{n_2+m_2} \dots p_k^{n_k+m_k}$.

(ii) a divides b if and only if $n_i \leq m_i$ for $i = 1, 2, \dots, k$.

(iii) $\gcd(a, b) = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$, where $s_i = \min(n_i, m_i)$.

(iv) $\mathrm{lcm}(a, b) = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$, where $t_i = \max(n_i, m_i)$.

Here $\mathrm{lcm}(a, b)$ denotes the **least common multiple** of a and b, that is, the smallest positive integer divisible by both a and b.

Problem. Are there positive integers a and b such that $\gcd(a^2, b^2) = 3$? Can we have $\gcd(a^2, b^2) = 8$?

Let $p_1 p_2 \dots p_k$ be the prime factorisation of a positive integer c. Then $p_1^2 p_2^2 \dots p_k^2$ is the prime factorisation of c^2. Hence each prime occurs in the prime factorisation of c^2 an even number of times.

It follows that whenever 3 is a common divisor of a^2 and b^2, so is $3^2 = 9$. Therefore $\gcd(a^2, b^2) \neq 3$.

Now suppose that a^2 and b^2 have common divisor $8 = 2^3$. Then a and b have common divisor $2^2 = 4$. Consequently, a^2 and b^2 have common divisor $4^2 = 16$ so that $\gcd(a^2, b^2) \neq 8$.

Remark. Note that $\gcd(a^2, b^2) = \big(\gcd(a, b)\big)^2$.