MATH 433

Applied Algebra

Lecture 26:
Review for Exam 2.

Topics for Exam 2

- Relations, properties of relations
- Finite state machines, automata

- Permutations
- Cycles, transpositions
- Cycle decomposition of a permutation
- Order of a permutation
- Sign of a permutation
- Symmetric and alternating groups

- Abstract groups (definition and examples)
- Basic properties of groups
- Semigroups
- Rings, zero-divisors
- Basic properties of rings
- Fields, characteristic of a field
- Vector spaces over a field

What you are supposed to remember

On permutations:

- Definition of a permutation, a cycle, and a transposition
- Theorem on cycle decomposition
- Definition of the order of a permutation
- How to find the order for a product of disjoint cycles
- Definition of even and odd permutations
- How to find the sign for a product of cycles

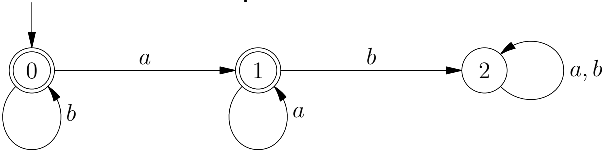
On algebraic structures:

- Definition of a group
- Definition of a semigroup
- Definition of a monoid
- Definition of a ring
- Definition of an integral domain
- Definition of a field
- Definition of a vector space over a field

Sample problems

Problem 1. Let R be a relation defined on the set of positive integers by xRy if and only if $\gcd(x, y) \neq 1$ ("is not coprime with"). Is this relation reflexive? Symmetric? Transitive?

Problem 2. A Moore diagram below depicts a 3-state acceptor automaton over the alphabet $\{a, b\}$ which accepts those input words that do not contain a subword ab (and rejects any input word containing a subword ab). Prove that no 2-state automaton can perform the same task.



Sample problems

Problem 3. List all cycles of length 3 in the symmetric group $S(4)$. Make sure there are no repetitions in your list.

Problem 4. Write the permutation $\pi = (4\ 5\ 6)(3\ 4\ 5)(1\ 2\ 3)$ as a product of disjoint cycles.

Problem 5. Find the order and the sign of the permutation $\sigma = (1\ 2)(3\ 4\ 5\ 6)(1\ 2\ 3\ 4)(5\ 6)$.

Problem 6. What is the largest possible order of an element of the alternating group $A(10)$?

Sample problems

Problem 7. Consider the operation $*$ defined on the set $\mathbb{Z}$ of integers by $a * b = a + b - 2$. Does this operation provide the integers with a group structure?

Problem 8. Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}$, where n and k are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does M form a field?

Sample problems

Problem 9. Let L be the set of the following 2×2 matrices with entries from the field $\mathbb{Z}_2$:

$$A = \begin{pmatrix} [0] & [0] \\ [0] & [0] \end{pmatrix}, \quad B = \begin{pmatrix} [1] & [0] \\ [0] & [1] \end{pmatrix},$$

$$C = \begin{pmatrix} [1] & [1] \\ [1] & [0] \end{pmatrix}, \quad D = \begin{pmatrix} [0] & [1] \\ [1] & [1] \end{pmatrix}.$$

Under the operations of matrix addition and multiplication, does this set form a ring? Does L form a field?

Problem 10. For any $\lambda \in \mathbb{Q}$ and any $v \in \mathbb{Z}$ let $\lambda \odot v = \lambda v$ if λv is an integer and $\lambda \odot v = v$ otherwise. Does this "selective scaling" make the additive Abelian group $\mathbb{Z}$ into a vector space over the field $\mathbb{Q}$?

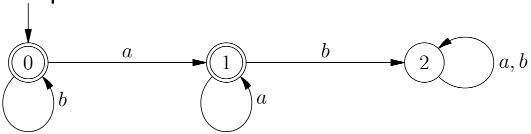
Problem 1. Let R be a relation defined on the set of positive integers by xRy if and only if $\gcd(x, y) \neq 1$ ("is not coprime with"). Is this relation reflexive? Symmetric? Transitive?

The relation R is not reflexive since 1 is not related to itself (actually, this is the only positive integer not related to itself by R).

The relation is symmetric since $\gcd(x, y) = \gcd(y, x)$ for all $x, y \in \mathbb{P}$.

The relation is not transitive as the following counterexample shows: $2R6$ and $6R3$, but 2 is not related to 3 by R.

Problem 2. A Moore diagram below depicts a 3-state acceptor automaton over the alphabet $\{a, b\}$ which accepts those input words that do not contain a subword ab. Prove that no 2-state automaton can perform the same task.



Assume the contrary: there is an automaton with two states 0 (initial) and 1 that does the job. We are going to reconstruct its transition function t.

Claim 1: $t(0, a) = 1$. Otherwise $t(0, a) = 0$, then we would not be able to distinguish inputs b and ab.

Claim 2: $t(0, b) = 0$. Otherwise $t(0, b) = 1$, then we would not be able to tell the input bb from ab.

Claim 3: $t(1, a) = 1$ (otherwise we would not tell b from aab).

Claim 4: $t(1, b) = 0$ (otherwise we would not tell aa from ab).

We still cannot distinguish bb from ab, a contradiction anyway.

Problem 3. List all cycles of length 3 in the symmetric group $S(4)$. Make sure there are no repetitions in your list.

Any cycle of length 3 in $S(4)$ moves 3 elements and fixes the remaining one. Therefore there are 4 ways to choose three elements a, b, c moved by such a cycle. For any choice of these, there are two cycles of length 3 moving a, b, c, each written in three different ways: $(a\ b\ c) = (b\ c\ a) = (c\ a\ b)$ and $(a\ c\ b) = (b\ a\ c) = (c\ b\ a)$.

The list: (1 2 3), (1 3 2), (1 2 4), (1 4 2), (1 3 4), (1 4 3), (2 3 4), (2 4 3).

Problem 4. Write the permutation $\pi = (4\ 5\ 6)(3\ 4\ 5)(1\ 2\ 3)$ as a product of disjoint cycles.

Keeping in mind that the composition is evaluated from the right to the left, we find that $\pi(1) = 2$, $\pi(2) = 5$, $\pi(5) = 3$, and $\pi(3) = 1$. Further, $\pi(4) = 6$ and $\pi(6) = 4$.
Thus $\pi = (1\ 2\ 5\ 3)(4\ 6)$.

Problem 5. Find the order and the sign of the permutation $\sigma = (1\ 2)(3\ 4\ 5\ 6)(1\ 2\ 3\ 4)(5\ 6)$.

First we find the cycle decomposition of the given permutation: $\sigma = (2\ 4)(3\ 5)$. It follows that the order of σ is 2 and that σ is an even permutation. Therefore the sign of σ is $+1$.

Problem 6. What is the largest possible order of an element of the alternating group $A(10)$?

The order of a permutation π is $o(\pi) = \mathrm{lcm}(l_1, l_2, \ldots, l_k)$, where $l_1, \ldots, l_k$ are lengths of cycles in the disjoint cycle decomposition of π.

The largest order for $\pi \in A(10)$, an even permutation of 10 elements, is 21. It is attained when π is the product of disjoint cycles of lengths 7 and 3, for example, $\pi = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ 10)$. One can check that in all other cases the order is at most 15.

Remark. The largest order for $\pi \in S(10)$ is $30 = 5 \cdot 3 \cdot 2$, but it is attained on odd permutations, e.g., $\pi = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8)(9\ 10)$.

Problem 7. Consider the operation $*$ defined on the set $\mathbb{Z}$ of integers by $a * b = a + b - 2$. Does this operation provide the integers with a group structure?

We need to check 4 axioms.

Closure: $a, b \in \mathbb{Z} \implies a * b = a + b - 2 \in \mathbb{Z}$.

Associativity: for any $a, b, c \in \mathbb{Z}$, we have
$(a*b)*c = (a+b-2)*c = (a+b-2)+c-2 = a+b+c-4$,
$a*(b*c) = a*(b+c-2) = a+(b+c-2)-2 = a+b+c-4$,
hence $(a * b) * c = a * (b * c)$.

Existence of identity: equalities $a * e = e * a = a$ are equivalent to $e + a - 2 = a$. They hold for $e = 2$.

Existence of inverse: equalities $a * b = b * a = e$ are equivalent to $b + a - 2 = e$ $(= 2)$. They hold for $b = 4 - a$.

Thus $(\mathbb{Z}, *)$ is a group.

Remark. Consider a bijection $f : \mathbb{Z} \to \mathbb{Z}$, $f(a) = a - 2$. Then $f(a * b) = f(a) + f(b)$ for all $a, b \in \mathbb{Z}$.

Problem 8. Let M be the set of all 2×2 matrices of the form $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}$, where n and k are rational numbers. Under the operations of matrix addition and multiplication, does this set form a ring? Does M form a field?

The set M is closed under matrix addition, taking the negative, and matrix multiplication as

$$\begin{pmatrix} n & k \\ 0 & n \end{pmatrix} + \begin{pmatrix} n' & k' \\ 0 & n' \end{pmatrix} = \begin{pmatrix} n+n' & k+k' \\ 0 & n+n' \end{pmatrix},$$

$$-\begin{pmatrix} n & k \\ 0 & n \end{pmatrix} = \begin{pmatrix} -n & -k \\ 0 & -n \end{pmatrix},$$

$$\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}\begin{pmatrix} n' & k' \\ 0 & n' \end{pmatrix} = \begin{pmatrix} nn' & nk'+kn' \\ 0 & nn' \end{pmatrix}.$$

Also, the multiplication is commutative on M. The associativity and commutativity of the addition, the associativity of the multiplication, and the distributive law hold on M since they hold for all 2×2 matrices. Thus M is a commutative ring.

The ring M has the unity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity matrix.

If a matrix $A \in M$ is invertible in the ring M and B is its inverse, then $AB = BA = I$. It follows that a matrix $A \in M$ is invertible in the ring M if and only if it is invertible in the sense of linear algebra and, moreover, the inverse matrix A^{-1} belongs to M.

A matrix $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix} \in M$ is invertible if $n \neq 0$, in which case

$$\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}^{-1} = \frac{1}{n^2} \begin{pmatrix} n & -k \\ 0 & n \end{pmatrix} = \begin{pmatrix} n^{-1} & -kn^{-2} \\ 0 & n^{-1} \end{pmatrix} \in M.$$

Since a nonzero matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M$ is not invertible, the ring M is not a field.

Problem 9. Let L be the set of the following 2×2 matrices with entries from the field $\mathbb{Z}_2$:

$$A=\begin{pmatrix}[0] & [0]\\ [0] & [0]\end{pmatrix},\ B=\begin{pmatrix}[1] & [0]\\ [0] & [1]\end{pmatrix},\ C=\begin{pmatrix}[1] & [1]\\ [1] & [0]\end{pmatrix},\ D=\begin{pmatrix}[0] & [1]\\ [1] & [1]\end{pmatrix}.$$

Under the operations of matrix addition and multiplication, does this set form a ring? Does L form a field?

First we build the addition and mutiplication tables for L (meanwhile checking that L is closed under both operations):

$+$	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

$\times$	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	D	B
D	A	D	B	C

Analyzing these tables, we find that both operations are commutative on L, A is the additive identity element, and B is the multiplicative identity element. Also, $B^{-1}=B$, $C^{-1}=D$, $D^{-1}=C$, and $-X=X$ for all $X\in L$. The associativity of addition and multiplication as well as the distributive law hold on L since they hold for all 2×2 matrices. Thus L is a field.

Problem 10. For any $\lambda \in \mathbb{Q}$ and any $v \in \mathbb{Z}$ let $\lambda \odot v = \lambda v$ if λv is an integer and $\lambda \odot v = v$ otherwise. Does this "selective scaling" make the additive Abelian group $\mathbb{Z}$ into a vector space over the field $\mathbb{Q}$?

The group $(\mathbb{Z}, +)$ with the scalar multiplication $\odot$ is not a vector space over $\mathbb{Q}$. One reason is that the distributive law $(\lambda + \mu) \odot v = \lambda \odot v + \mu \odot v$ does not hold.

A counterexample is $\lambda = \mu = 1/2$ and $v = 1$. Then $(\frac{1}{2} + \frac{1}{2}) \odot v = 1 \odot v = v = 1$ while $\frac{1}{2} \odot v + \frac{1}{2} \odot v = v + v = 2$.

Remark. The essential information about the scalar multiplication $\odot$ used in the above counterexample is that $1 \odot v = v$ and $\frac{1}{2} \odot v$ is an integer. It follows that the additive group $\mathbb{Z}$, in principle, cannot be made into a vector space over $\mathbb{Q}$.