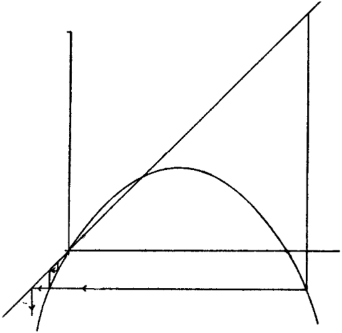
MATH 614

Dynamical Systems and Chaos

Lecture 4:
Itineraries.
Cantor sets.

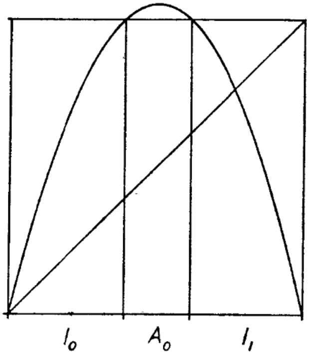
Logistic map

The **logistic map** is any of the family of quadratic maps $F_\mu(x) = \mu x(1-x)$ depending on the parameter $\mu \in \mathbb{R}$.



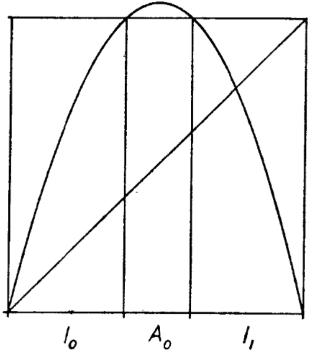
If $\mu > 1$, then for any $x < 0$ the orbit $x, F_\mu(x), F_\mu^2(x), \dots$ is decreasing and diverges to $-\infty$. Besides, the interval $(1,\infty)$ is mapped onto $(-\infty, 0)$. Hence all nontrivial dynamics (if any) is concentrated on the interval $I = [0,1]$.

Logistic map: $\mu > 4$



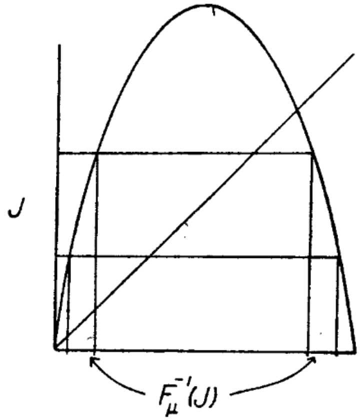
The interval $I = [0, 1]$ is invariant under the map F_μ for $0 \le \mu \le 4$. In the case $\mu > 4$, this interval splits into 3 subintervals: $I = I_0 \cup A_0 \cup I_1$, where closed intervals $I_0 = [0, x_0]$ and $I_1 = [x_1, 1]$ are mapped monotonically onto I while an open interval $A_0 = (x_0, x_1)$ is mapped onto $(1, \mu/4]$.

Logistic map: $\mu > 4$



The splitting points satisfy $F_\mu(x_0) = F_\mu(x_1) = 1$. Since $F_\mu(x) = 1 \iff \mu x(1-x) = 1 \iff x^2 - x + \mu^{-1} = 0$,

we obtain $x_0 = \dfrac{1}{2} - \sqrt{\dfrac{1}{4} - \dfrac{1}{\mu}}, \quad x_1 = \dfrac{1}{2} + \sqrt{\dfrac{1}{4} - \dfrac{1}{\mu}}.$



For any interval $J \subset I$, the preimage $F_\mu^{-1}(J)$ consists of two intervals $J_0 \subset I_0$ and $J_1 \subset I_1$. Each of the intervals J_0 and J_1 is mapped monotonically onto J. If the interval J is closed (resp. open), then so are the intervals J_0 and J_1.

Let us define sets $A_1, A_2, \ldots$ inductively by $A_n = F_\mu^{-1}(A_{n-1})$, $n = 1, 2, \ldots$ Then A_1 consists of two disjoint open intervals, A_2 consists of 4 disjoint open intervals, and so on. In general, the set A_n consists of 2^n disjoint open intervals.

It follows by induction on n that $A_n = \{x \in I \mid F_\mu^n(x) \in A_0\}$, $n = 0, 1, 2, \ldots$ As a consequence, the sets $A_0, A_1, A_2, \ldots$ are disjoint from each other.

Let $\Lambda = I \setminus (A_0 \cup A_1 \cup A_2 \cup \ldots)$. Then Λ is the set of all points $x \in \mathbb{R}$ such that the orbit $O^+(x)$ is contained in I. Notice that $F_\mu(\Lambda) \subset \Lambda$. Hence the restriction of the map F_μ to the set Λ defines a new dynamical system.

Itineraries

Any element of the set Λ belongs to either I_0 or I_1. For any $x \in \Lambda$ let $S(x) = (s_0 s_1 s_2 \dots)$ be an infinite sequence of 0's and 1's defined so that $F_\mu^n(x) \in I_{s_n}$ for $n = 0, 1, 2, \dots$ The sequence $S(x)$ is called the **itinerary** of the point x.

Let Σ_2 denote the set of all infinite sequences of 0's and 1's. Then the itinerary can be regarded as a map $S : \Lambda \to \Sigma_2$. If $S(x) = (s_0 s_1 s_2 \dots)$, then $S(F_\mu(x)) = (s_1 s_2 \dots)$. Therefore we have a commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{F_\mu} & \Lambda \\ S \big\downarrow & & \big\downarrow S \\ \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \end{array}$$

that is, $S \circ F_\mu = \sigma \circ S$, where $\sigma : \Sigma_2 \to \Sigma_2$ is a transformation defined by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$. This transformation is called the **shift**.

Now we are going to define a closed interval $I_{s_0 s_1 \dots s_n} \subset I$ for any finite sequence $s_0 s_1 \dots s_n$ of 0's an 1's. The intervals I_0 and I_1 are already defined. The others are defined inductively (induction on the length of the sequence):

$$I_{s_0 s_1 \dots s_n} = I_{s_0} \cap F_\mu^{-1}(I_{s_1 \dots s_n}).$$

These intervals have the following properties:

- F_μ maps $I_{s_0 s_1 \dots s_n}$ monotonically onto $I_{s_1 \dots s_n}$;
- F_μ^{n+1} maps $I_{s_0 s_1 \dots s_n}$ monotonically onto I;
- $I_{s_0 s_1 \dots s_n}$ consists of all points x such that $x \in I_{s_0}$, $F_\mu(x) \in I_{s_1}$, $F_\mu^2(x) \in I_{s_2}$, ..., $F_\mu^n(x) \in I_{s_n}$;
- intervals $I_{s_0 s_1 \dots s_n}$ and $I_{t_0 t_1 \dots t_n}$ are disjoint if the sequences $s_0 s_1 \dots s_n$ and $t_0 t_1 \dots t_n$ are not the same;
- for any infinite sequence $(s_0 s_1 s_2 \dots) \in \Sigma_2$, the intervals $I_{s_0}, I_{s_0 s_1}, I_{s_0 s_1 s_2}, \dots$ are nested, i.e., $I_{s_0 s_1 \dots s_n s_{n+1}} \subset I_{s_0 s_1 \dots s_n}$ for $n = 0, 1, 2, \dots$;
- for any infinite sequence $\mathbf{s} = (s_0 s_1 s_2 \dots) \in \Sigma_2$, $S^{-1}(\mathbf{s}) = I_{s_0} \cap I_{s_0 s_1} \cap I_{s_0 s_1 s_2} \cap \dots$

By the above for any infinite sequence $\mathbf{s} \in \Sigma_2$ the preimage $S^{-1}(\mathbf{s})$ is the intersection of nested closed intervals. Therefore $S^{-1}(\mathbf{s})$ is either a point or a closed interval. In particular, the preimage is never empty so that the itinerary map S is onto.

The construction of the set Λ and the itinerary map $S : \Lambda \to \Sigma_2$ can be performed for maps more general than the logistic map F_μ, $\mu > 4$. Namely, it is enough to consider any continuous map $f : I \to \mathbb{R}$ satisfying the following properties:

- $f(0) = f(1) = 0$;
- there exists a point $x_{\max} \in (0, 1)$ such that f is strictly increasing on $[0, x_{\max}]$ and strictly decreasing on $[x_{\max}, 1]$;
- $f(x_{\max}) > 1$.

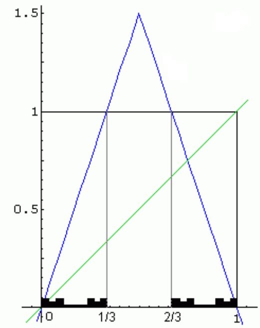
Although the itinerary map is always onto, it need not be one-to-one.

Tent map

The **tent map** is any of a family of piecewise linear maps

$$T_\mu(x) = \mu \min(x, 1-x) = \begin{cases} \mu x & \text{if } x < 1/2, \\ \mu(1-x) & \text{if } x \geq 1/2 \end{cases}$$

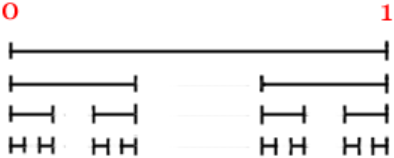
depending on the parameter $\mu \in \mathbb{R}$.



The set $\Lambda = \Lambda_\mu$ and the itinerary map $S = S_\mu$ can be constructed when $\mu > 2$.

Cantor set

In the case $\mu = 3$, the interval A_0 is the middle third of I so that Λ_3 is the **Cantor Middle-Thirds Set**.



The set Λ_3 consists of those points $x \in [0, 1]$ that admit a ternary expansion $0.s_1 s_2 \ldots$ without any 1's (only 0's and 2's), in which case $S_3(x) = (\bar{s}_1 \bar{s}_2 \ldots)$, where $\bar{0} = 0$ and $\bar{2} = 1$.

For any $\mu > 2$, the set Λ_μ is a fractal set of dimension $\log_\mu 2 < 1$.

General Cantor sets

Definition. A subset Λ of the real line $\mathbb{R}$ is called a (general) **Cantor set** if it is

- nonempty,
- bounded,
- closed,
- totally disconnected, which means that Λ contains no intervals, and
- perfect, which means that Λ has no isolated points.

Theorem Any two Cantor sets are homeomorphic. That is, if Λ and Λ' are Cantor sets, then there exists a homeomorphism $\phi : \Lambda \to \Lambda'$.

Furthermore, the homeomorphism ϕ can be chosen strictly increasing, in which case it can be extended to a homeomorphism $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$.