MATH 614

Dynamical Systems and Chaos

Lecture 6: Symbolic dynamics (continued). Topological conjugacy. Definition of chaos.

Interior and boundary

Let X be a topological space. Any open set of the topology containing a point $x \in X$ is called a **neighborhood** of x.

Let E be a subset of X. A point $x \in E$ is called an **interior point** of E if some neighborhood of x is contained in E. The set of all interior points of E is called the **interior** of E and denoted $\mathrm{int}(E)$.

A point $x \in X$ is called a **boundary point** of the set E if each neighborhood of x intersets both E and $X \setminus E$ (the point x need not belong to E). The set of all boundary points of E is called the **boundary** of E and denoted ∂E.

The union $E \cup \partial E$ is called the **closure** of E and denoted $\overline{E}$. The set E is called **closed** if $\overline{E} = E$.

Let E be an arbitrary subset of the topological space X.

Proposition 1 The topological space X is the disjoint union of three sets: $X = \mathrm{int}(E) \cup \partial E \cup \mathrm{int}(X \setminus E)$.

Proposition 2 The set E is closed if and only if the complement $X \setminus E$ is open.

Proposition 3 The interior $\mathrm{int}(E)$ is the largest open subset of E.

Proposition 4 The closure $\overline{E}$ is the smallest closed set containing E.

Definition. We say that a subset $E \subset X$ is **dense** in X if $\overline{E} = X$. An equivalent condition is that E intersects every nonempty open set. We say that E is **dense in a set** $U \subset X$ if the set U is contained in $\overline{E \cap U}$.

Symbolic dynamics

Given a finite set $\mathcal{A}$ (an alphabet), we denote by $\Sigma_{\mathcal{A}}$ the set of all infinite words over $\mathcal{A}$, i.e., infinite sequences $\mathbf{s} = (s_1 s_2 \dots)$, $s_i \in \mathcal{A}$.

For any finite word w over the alphabet $\mathcal{A}$, that is, $w = s_1 s_2 \dots s_n$, $s_i \in \mathcal{A}$, we define a **cylinder** $C(w)$ to be the set of all infinite words $\mathbf{s} \in \Sigma_{\mathcal{A}}$ that begin with w. The topology on $\Sigma_{\mathcal{A}}$ is defined so that open sets are unions of cylinders. Two infinite words are considered close in this topology if they have a long common beginning.

The **shift** transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is defined by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 \dots)$. This transformation is continuous. The study of the shift and related transformations is called **symbolic dynamics**.

Periodic points of the shift

Definition (corrected). A point $x \in X$ is a **periodic** point of **period** n of a map $f : X \to X$ if $f^n(x) = x$. The least $n \geq 1$ satisfying this relation is called the **prime period** of x.

Suppose $\mathbf{s} \in \Sigma_{\mathcal{A}}$. Given a natural number n, let $\mathbf{s}' = \sigma^n(\mathbf{s})$ and w be the beginning of length n of $\mathbf{s}$. Then $\mathbf{s} = w\mathbf{s}'$. It follows that $\sigma^n(\mathbf{s}) = \mathbf{s}$ if and only if $\mathbf{s} = www\dots$ Similarly, an infinite word $\mathbf{t}$ is an eventually periodic point of the shift if and only if $\mathbf{t} = uwww\dots$ for some finite words u and w.

Proposition (i) The number of periodic points of period n is k^n, where k is the number of elements in the alphabet $\mathcal{A}$.
(ii) Periodic points are dense in $\Sigma_{\mathcal{A}}$.

Proof: By the above the number of periodic points of period n equals the number of finite words of length n, which is k^n. Further, any cylinder $C(w)$ contains a periodic point $www\dots$ Consequently, any open set $U \subset \Sigma_{\mathcal{A}}$ contains a periodic point.

Dense orbit of the shift

Proposition The shift transformation $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ admits a dense orbit.

Proof: Since open subsets of $\Sigma_{\mathcal{A}}$ are unions of cylinders, it follows that a set $E \subset \Sigma_{\mathcal{A}}$ is dense if and only if it intersects every cylinder.

The orbit under the shift of an infinite word $\mathbf{s} \in \Sigma_{\mathcal{A}}$ visits a particular cylinder $C(w)$ if and only if the finite word w appears somewhere in $\mathbf{s}$, that is, $\mathbf{s} = w_0 w \mathbf{s}_0$, where w_0 is a finite word and $\mathbf{s}_0$ is an infinite word. Therefore the orbit $O_\sigma^+(\mathbf{s})$ is dense in $\Sigma_{\mathcal{A}}$ if and only if the infinite word $\mathbf{s}$ contains all finite words over the alphabet $\mathcal{A}$ as subwords.

There are only countably many finite words over $\mathcal{A}$. We can enumerate them all: $w_1, w_2, w_3, \ldots$ Then an infinite word $\mathbf{s} = w_1 w_2 w_3 \ldots$ has dense orbit.

Subshift

Suppose Σ' is a closed subset of the space $\Sigma_{\mathcal{A}}$ invariant under the shift σ, i.e., $\sigma(\Sigma') \subset \Sigma'$. The restriction of the shift σ to the set Σ' is called a **subshift**.

Examples. • Orbit closure $\overline{O_\sigma^+(\mathbf{s})}$ is always shift-invariant.

• Let $\mathcal{A} = \{0, 1\}$ and Σ' consists of $(00\ldots)$, $(11\ldots)$, and all sequences of the form $(0\ldots 011\ldots)$. Then Σ' is a closed, shift-invariant set that is not an orbit closure.

• Suppose W is a collection of finite words in the alphabet $\mathcal{A}$. Let Σ' be the set of all $\mathbf{s} \in \Sigma_{\mathcal{A}}$ that do not contain any element of W as a subword. Then Σ' is a closed, shift-invariant set. Any subshift can be defined this way. In the previous example, $W = \{10\}$.

• In the case the set W of "forbidden" words is finite, the subshift is called a **subshift of finite type**.

Random dynamical system

Let f_0 and f_1 be two transformations of a set X. Consider a random dynamical system $F : X \to X$ defined by $F(x) = f_\xi(x)$, where ξ is a random variable taking values 0 and 1.

The symbolic dynamics allows to redefine this dynamical system as a deterministic one. The phase space of the new system is $X \times \Sigma_{\{0,1\}}$ and the transformation is given by

$$\mathcal{F}(x, \mathbf{s}) = \big(f_{s_1(\mathbf{s})}(x), \sigma(\mathbf{s})\big),$$

where $s_1(\mathbf{s})$ is the first entry of the sequence $\mathbf{s}$.

Topological conjugacy

Suppose $f : X \to X$ and $g : Y \to Y$ are transformations of topological spaces.

Definition. We say that a map $\phi : X \to Y$ is a **semi-conjugacy** of f with g if ϕ is onto and $\phi \circ f = g \circ \phi$.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \phi \downarrow & & \downarrow \phi \\ Y & \xrightarrow{g} & Y \end{array}$$

The map ϕ is a **conjugacy** if, additionally, it is invertible.
The map ϕ is a **topological conjugacy** if, additionally, it is a homeomorphism, which means that both ϕ and ϕ^{-1} are continuous.

In the latter case, we say that the maps f and g are **topologically conjugate**.

Examples of topological conjugacy

- Linear maps $f(x) = \lambda x$ and $g(x) = \mu x$ on $\mathbb{R}$ are topologically conjugate if $0 < \lambda, \mu < 1$ or if $\lambda, \mu > 1$. If $0 < \lambda < 1 < \mu$, then they are not topologically conjugate.

- The maps $f(x) = x/2$, $g(x) = x^3$, and $h(x) = x - x^3$ are topologically conjugate on $[-1/2, 1/2]$.

- Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map and Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0,1]$. If the itinerary map $S : \Lambda \to \Sigma_{\{0,1\}}$ is one-to-one, then it provides topological conjugacy of the restriction $f|_\Lambda$ of the map f to Λ with the shift $\sigma : \Sigma_{\{0,1\}} \to \Sigma_{\{0,1\}}$. In general, S is a continuous semi-conjugacy.

Topological transitivity

Suppose $f : X \to X$ is a continuous transformation of a topological space X.

Definition. The map f is **topologically transitive** if for any nonempty open sets $U, V \subset X$ there exists a natural number n such that $f^n(U) \cap V \neq \emptyset$.

Topological transitivity means that the dynamical system f is, in a sense, indecomposable. One sufficient condition of topological transitivity is the existence of a dense orbit. If the space X is compact, then this condition is necessary as well.

It is easy to see that topological transitivity is preserved under topological conjugacy.

Separation of orbits

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that the map f has **sensitive dependence on initial conditions** if there is $\delta > 0$ such that, for any $x \in X$ and a neighborhood U of x, there exist $y \in U$ and $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

We say that the map f is **expansive** if there is $\delta > 0$ such that, for any $x, y \in X$, $x \neq y$, there exists $n \geq 0$ satisfying $d\big(f^n(y), f^n(x)\big) > \delta$.

Obviously, expansiveness implies sensitive dependence on initial conditions.

Definition of chaos

Suppose $f : X \to X$ is a continuous transformation of a metric space (X, d).

Definition. We say that the map f is **chaotic** if

- f has sensitive dependence on initial conditions;
- f is topologically transitive;
- periodic points of f are dense in X.

The three conditions provide the dynamical system f with unpredictability, indecomposability, and an element of regularity (recurrence).

Examples of chaotic systems

- The shift $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ is chaotic.

- Let $f : \mathbb{R} \to \mathbb{R}$ be a unimodal map and Λ be the set of all points $x \in \mathbb{R}$ such that $O_f^+(x) \subset [0, 1]$. If Λ is a Cantor set then the restriction $f|_\Lambda$ of the map f to Λ is chaotic (otherwise it is not).

Recall that Λ is a Cantor set if and only if the itinerary map $S : \Lambda \to \Sigma_{\{0,1\}}$ is one-to-one, in which case S is a topological conjugacy of $f|_\Lambda$ with the shift on $\Sigma_{\{0,1\}}$.