MATH 614

Dynamical Systems and Chaos

Lecture 8: Structural stability. Sharkovskii's theorem.

Structural stability

Informally, a dynamical system is **structurally stable** if its structure is preserved under small perturbations. To make this notion formal, one has to specify what it means that the "structure is preserved" and what is considered a "small perturbation".

In the context of topological dynamics, structural stability usually means that the perturbed system is topologically conjugate to the original one.

The description of small perturbations varies for different dynamical systems and so there are various kinds of structural stability.

- Structural stability within a parametric family.

Suppose $f_{\mathbf{p}} : X_{\mathbf{p}} \to X_{\mathbf{p}}$ is a dynamical system depending on a parameter vector $\mathbf{p} \in P$, where $P \subset \mathbb{R}^k$. Given $\mathbf{p}_0 \in P$, we say that $f_{\mathbf{p}_0}$ is **structurally stable within the family** $\{f_{\mathbf{p}}\}$ if there exists $\varepsilon > 0$ such that for any $\mathbf{p} \in P$ satisfying $|\mathbf{p} - \mathbf{p}_0| < \varepsilon$ the system $f_{\mathbf{p}}$ is topologically conjugate to $f_{\mathbf{p}_0}$.

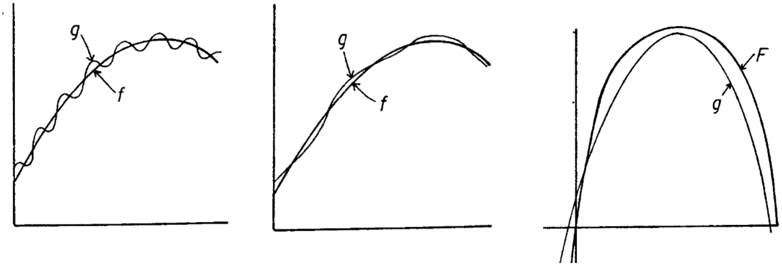
- C^r-structural stability for one-dimensional systems.

Let J be an interval of the real line. For any integer $r \geq 0$, let $C^r(J)$ denote the set of r times continuously differentiable functions $f : J \to \mathbb{R}$. The C^r distance between functions $f, g \in C^r(J)$ is given by

$$d_r(f,g) = \sup_{x \in J} \big(|f(x) - g(x)|, |f'(x) - g'(x)|, \ldots, |f^{(r)}(x) - g^{(r)}(x)|\big).$$

We say that a map $f \in C^r(J)$ is C^r-**structurally stable** if there exists $\varepsilon > 0$ such that whenever $d_r(f,g) < \varepsilon$, it follows that g is topologically conjugate to f.

Small perturbation

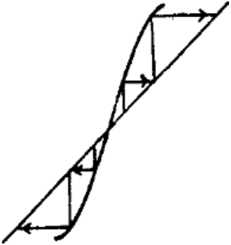


In the first figure, the function g is a C^0-small perturbation of f, but not a C^1-small one. In the second figure, the functions f and g are C^1-close but not C^2-close. In the third figure, f and g are C^2-close.

Examples of structural stability

- Linear map $f_\lambda : \mathbb{R} \to \mathbb{R}$, $f_\lambda(x) = \lambda x$.

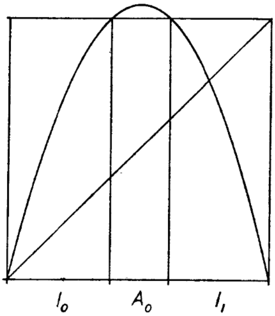
The map f_λ is structurally stable within the family $\{f_\lambda\}$ if and only if $\lambda \notin \{-1, 0, 1\}$. Besides, it is C^1-structurally stable for the same values of λ.

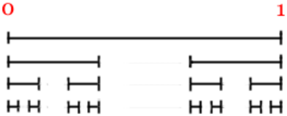


Examples of structural stability

- Logistic map $F_\mu : \mathbb{R} \to \mathbb{R}$, $F_\mu(x) = \mu x(1-x)$.

The map F_μ is structurally stable within the family $\{F_\mu\}$ for $\mu > 4$. Besides, it is C^2-structurally stable for $\mu > 4$ (but not C^1-structurally stable).





Period set

Suppose J is an interval of the real line and $f : J \to J$ is a continuous map. Let $\mathcal{P}(f)$ be the set of all natural numbers n for which the map f admits a periodic point of prime period n (or, equivalently, a periodic orbit that consists of n points).

Question. Which subsets of $\mathbb{N}$ can occur as $\mathcal{P}(f)$?

Examples. • $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$.
$\mathcal{P}(f) = \emptyset$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$.
$\mathcal{P}(f) = \{1\}$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x$.
$\mathcal{P}(f) = \{1, 2\}$.

• $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \mu x(1 - x)$, where $\mu > 4$.
The map f has an invariant set Λ such that the restriction $f|_\Lambda$ is conjugate to the shift on $\Sigma_{\{0,1\}}$. Since the shift admits periodic points of all prime periods, so does f: $\mathcal{P}(f) = \mathbb{N}$.

Sharkovskii's ordering

The **Sharkovskii ordering** is the following strict linear ordering of the natural numbers:

$$\begin{array}{cccccccccc}
 & 3 & \rhd & 5 & \rhd & 7 & \rhd & 9 & \rhd & \dots \\
\rhd & 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & 2\cdot 9 & \rhd & \dots \\
\rhd & 2^2\cdot 3 & \rhd & 2^2\cdot 5 & \rhd & 2^2\cdot 7 & \rhd & 2^2\cdot 9 & \rhd & \dots
\end{array}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots \rhd 2^k \rhd \dots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

To be precise, for any integers $k_1, k_2 \geq 0$ and odd natural numbers p_1, p_2 we let $2^{k_1}p_1 \rhd 2^{k_2}p_2$ if and only if one of the following conditions holds:

- $k_1 = k_2$ and $1 < p_1 < p_2$;
- $p_1, p_2 > 1$ and $k_1 < k_2$;
- $p_1 > 1$ and $p_2 = 1$;
- $p_1 = p_2 = 1$ and $k_1 > k_2$.

Sharkovskii's Theorem

Theorem 1 (Sharkovskii) Suppose $f : J \to J$ is a continuous map of an interval $J \subset \mathbb{R}$. If f admits a periodic point of prime period n and $n \triangleright m$ for some $m \in \mathbb{N}$, then f admits a periodic point of prime period m as well.

Definition. A subset $E \subset \mathbb{N}$ is called a **tail** of Sharkovskii's ordering if $n \in E$ and $n \triangleright m$ implies $m \in E$ for all $m, n \in \mathbb{N}$.

Sharkovskii's Theorem states that the period set $\mathcal{P}(f)$ is such a tail. For any $n \in \mathbb{N}$ the set $E_n = \{n\} \cup \{m \in \mathbb{N} \mid n \triangleright m\}$ is a tail. The only tails that cannot be represented this way are $\{2^n \mid n \geq 0\}$ and the empty set.

Theorem 2 For any tail E of Sharkovskii's ordering there exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{P}(f) = E$.

Remark. For maps of an interval $J \subset \mathbb{R}$, Theorem 2 holds with one exception: if J is bounded and closed, then $\mathcal{P}(f) \neq \emptyset$.

Suppose $f : J \to J$ is a continuous map of an interval $J \subset \mathbb{R}$. Given two closed bounded intervals $I_1, I_2 \subset J$, we write $\boxed{I_1 \to I_2}$ if $f(I_1) \supset I_2$ (i.e., if I_1 covers I_2 under the action of f).

Lemma 1 If $I \to I$, then the interval I contains a fixed point of the map f.

Proof: Let $I = [a, b]$. Since $f(I) \supset I$, there exist $a_0, b_0 \in I$ such that $f(a_0) = a$, $f(b_0) = b$. Then a continuous function $g(x) = f(x) - x$ satisfies $g(a_0) = a - a_0 \leq 0$ and $g(b_0) = b - b_0 \geq 0$. By the Intermediate Value Theorem, we have $g(c) = 0$ for some c between a_0 and b_0. Then $c \in I$ and $f(c) = c$.

Lemma 2 If the map f has a periodic orbit, then it has a fixed point.

Proof: Suppose x is a periodic point of f of prime period n. In the case $n = 1$, we are done. Otherwise let $x_1, x_2, \ldots, x_n$ be the list of all points of the orbit $O_f^+(x)$ ordered so that $x_1 < x_2 < \cdots < x_n$. Note that $f(x_i) \neq x_i$ for all i. In particular, $f(x_1) > x_1$ while $f(x_n) < x_n$.

Let j be the largest index satisfying $f(x_j) > x_j$. Then $j < n$, $f(x_j) \geq x_{j+1}$, and $f(x_{j+1}) \leq x_j$. The Intermediate Value Theorem implies that $[x_j, x_{j+1}] \to [x_j, x_{j+1}]$. By Lemma 1, the map f has a fixed point in the interval $[x_j, x_{j+1}]$.

Lemma 3 If $I \to I'$, then there exists a closed interval $I_0 \subset I$ such that f maps I_0 onto I'.

Proof: Let $I' = [a, b]$. Then $A = I \cap f^{-1}(a)$ and $B = I \cap f^{-1}(b)$ are nonempty compact sets. It follows that the distance function $d(x, y) = |y - x|$ attains its minimum on the set $A \times B$ at some point (x_0, y_0). Note that $x_0 \neq y_0$ since $A \cap B = \emptyset$. Let I_0 denote the closed interval with endpoints x_0 and y_0. Then $I_0 \subset I$, the endpoints of I_0 are mapped to a and b, and no interior point of I_0 is mapped to a or b. The Intermediate Value Theorem implies that $f(I_0) = I'$.

Lemma 4 If $I_1 \to I_2 \to \cdots \to I_n \to I_1$, then there exists a fixed point x of f^n such that $x \in I_1$, $f(x) \in I_2$, ..., $f^{n-1}(x) \in I_n$.

Proof: It follows by induction from Lemma 3 that there exist closed intervals $I_1' \subset I_1$, $I_2' \subset I_2$, ..., $I_n' \subset I_n$ such that f maps I_i' onto I_{i+1}' for $1 \leq i \leq n-1$ and also maps I_n' onto I_1. As a consequence, f^n maps I_1' onto I_1. Lemma 1 implies that f^n has a fixed point $x \in I_1'$. By construction, $f^i(x) \in I_i' \subset I_i$ for $0 \leq i \leq n-1$.

Proposition 5 If the map f has a periodic point of prime period 3, then it has periodic points of any prime period.

Proof: Suppose x_1, x_2, x_3 are points forming a periodic orbit of f, ordered so that $x_1 < x_2 < x_3$. We have that either $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1$, or else $f(x_1) = x_3$, $f(x_2) = x_1$, $f(x_3) = x_2$. In the first case, let $I_1 = [x_2, x_3]$ and $I_2 = [x_1, x_2]$. Otherwise we let $I_1 = [x_1, x_2]$ and $I_2 = [x_2, x_3]$. Then $I_1 \to I_2 \to I_1$ and $I_1 \to I_1$.

The map f has a periodic point of prime period 3. By Lemma 2, it also has a fixed point. To find a periodic point of prime period n, where $n = 2$ or $n \geq 4$, we notice that

$$I_2 \to \underbrace{I_1 \to I_1 \to \cdots \to I_1}_{n-1 \text{ times}} \to I_2.$$

By Lemma 4, there exists $x \in I_2$ such that $f^n(x) = x$ and $f^i(x) \in I_1$ for $1 \leq i \leq n-1$. If $x \notin I_1$, we obtain that n is the prime period of x. Otherwise $x = x_2$, which leads to a contradiction.