MATH 614

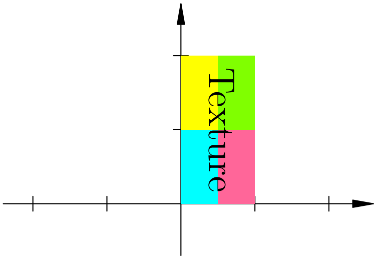
Dynamical Systems and Chaos

Lecture 13:
Dynamics of linear maps.
Hyperbolic toral automorphisms.

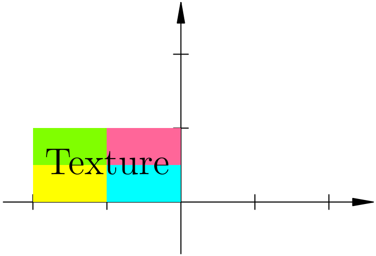
Linear transformations

Any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is represented as multiplication of an n-dimensional column vector by a $n \times n$ matrix: $L(\mathbf{x}) = A\mathbf{x}$, where $A = (a_{ij})_{1 \le i,j \le n}$.

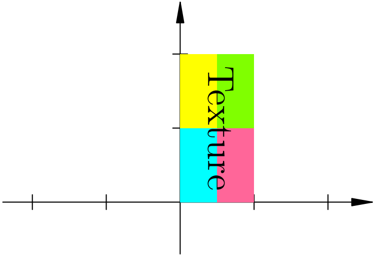
Dynamics of linear transformations corresponding to particular matrices is determined by eigenvalues and the Jordan canonical form.



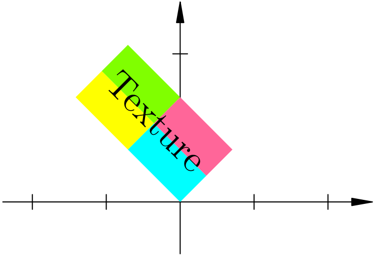
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



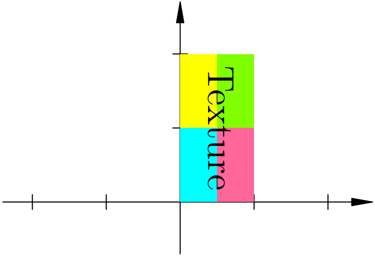
Rotation by 90^{o}



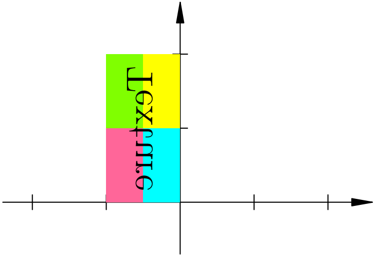
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$



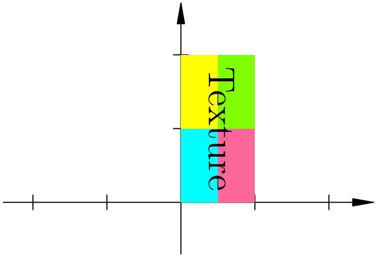
Rotation by 45^{o}



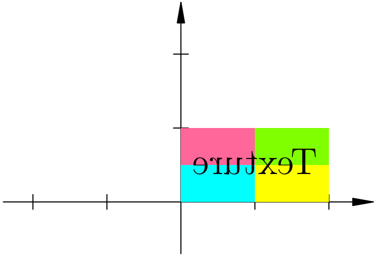
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



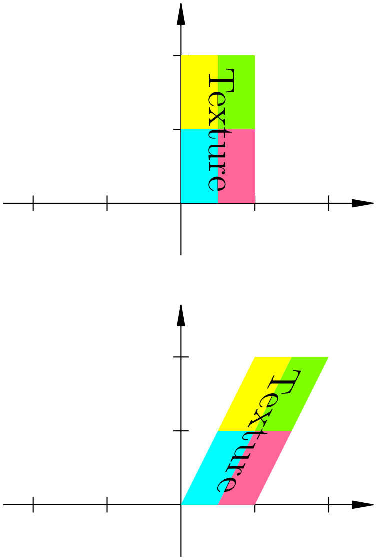
Reflection about
the vertical axis



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

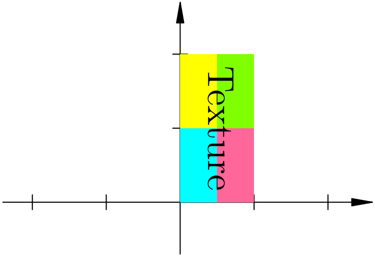


Reflection about
the line $x - y = 0$

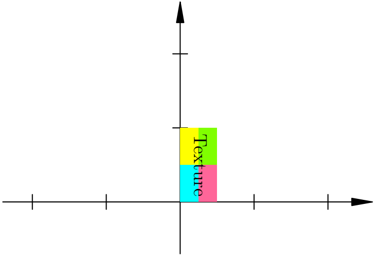


$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

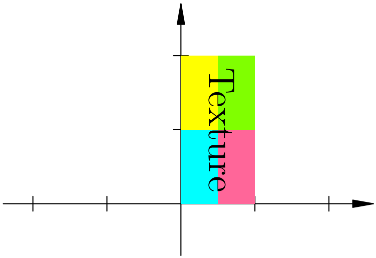
Horizontal shear



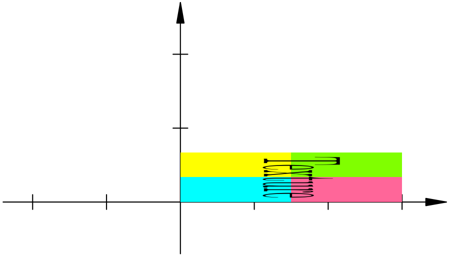
$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$



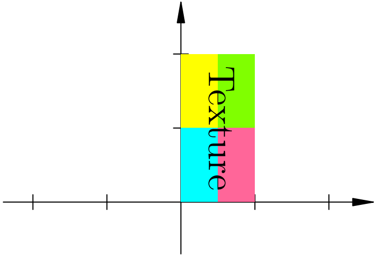
Scaling



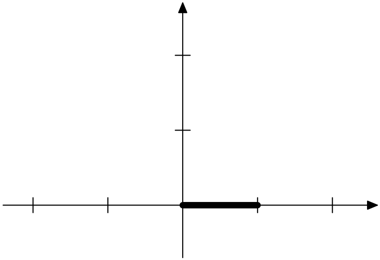
$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$



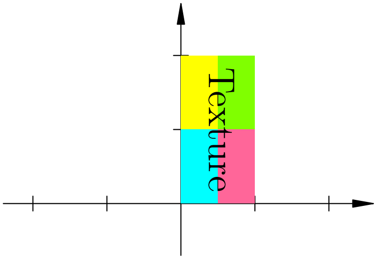
Squeeze



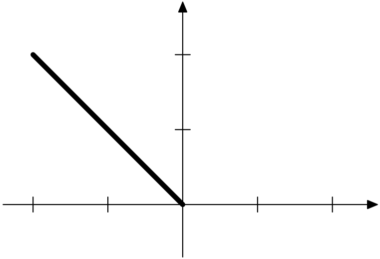
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



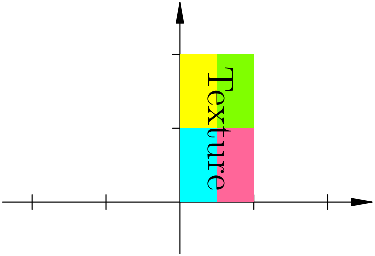
Vertical projection on the horizontal axis



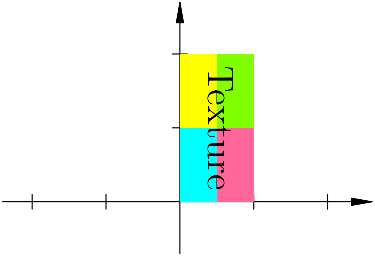
$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$



Horizontal projection
on the line $x + y = 0$



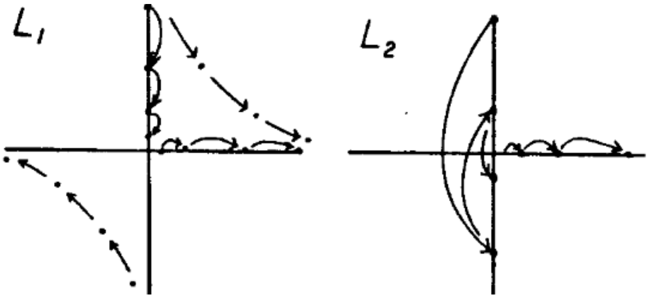
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Identity

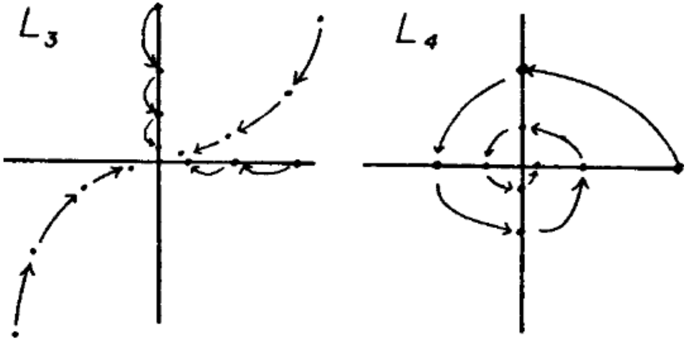
Phase portraits of linear maps

$$L_1(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \mathbf{x} \qquad L_2(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & -1/2 \end{pmatrix} \mathbf{x}$$



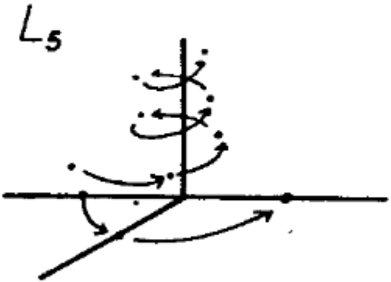
Phase portraits of linear maps

$$L_3(\mathbf{x}) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \mathbf{x} \qquad L_4(\mathbf{x}) = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \mathbf{x}$$



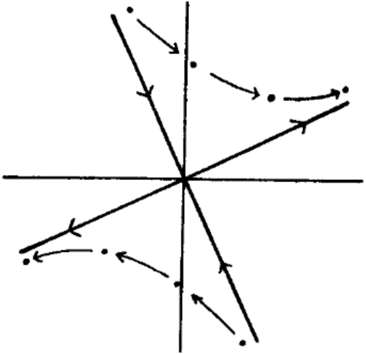
Phase portraits of linear maps

$$L_5(\mathbf{x}) = \begin{pmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$$



Phase portraits of linear maps

$$L(\mathbf{x}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x}$$



Stable and unstable subspaces

Proposition 1 Suppose that all eigenvalues of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ are less than 1 in absolute value. Then $L^n(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proposition 2 Suppose that all eigenvalues of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ are greater than 1 in absolute value. Then $L^{-n}(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$ for all $\mathbf{x} \in \mathbb{R}^n$.

Given a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$, let W^s denote the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $L^n(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$. In the case L is invertible, let W^u denote the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $L^{-n}(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$.

Proposition 3 W^s and W^u are vector subspaces of $\mathbb{R}^n$ that are transversal: $W^s \cap W^u = \{\mathbf{0}\}$.

Definition. W^s is called the **stable subspace** of the linear map L. W^u is called the **unstable subspace** of L.

Hyperbolic linear maps

Definition. A linear map L is called **hyperbolic** if it is invertible and all eigenvalues of L are different from 1 in absolute value.

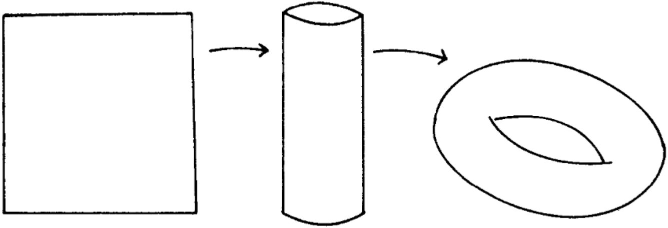
Proposition Suppose $L : \mathbb{R}^n \to \mathbb{R}^n$ is a hyperbolic linear map. Then

- $W^s \oplus W^u = \mathbb{R}^n$;
- if $\mathbf{x} \notin W^s \cup W^u$, then $L^n(\mathbf{x}) \to \infty$ as

$n \to \pm\infty$.

Torus

The two-dimensional **torus** is a closed surface obtained by gluing together opposite sides of a square by translation.



Torus

The **two-dimensional torus** is a closed surface obtained by gluing together opposite sides of a square by translation.

Alternatively, the torus is defined as $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the quotient of the plane $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$. To be precise, we introduce a relation on $\mathbb{R}^2$: $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in \mathbb{Z}^2$. This is an equivalence relation and $\mathbb{T}^2$ is the set of equivalence classes. The plane $\mathbb{R}^2$ induces a distance function, a topology, and a smooth structure on the torus $\mathbb{T}^2$. Also, the addition is well defined on $\mathbb{T}^2$. We denote the equivalence class of a point $(x, y) \in \mathbb{R}^2$ by $[x, y]$.

Topologically, the torus $\mathbb{T}^2$ is the Cartesian product of two circles: $\mathbb{T}^2 = S^1 \times S^1$.

Similarly, the *n*-**dimensional torus** is defined as $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Topologically, it is the Cartesian product of n circles: $\mathbb{T}^n = S^1 \times \cdots \times S^1$.

Transformations of the torus

Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the natural projection, $\pi(x_1, \dots, x_n) = [x_1, \dots, x_n]$. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation such that $\mathbf{x} \sim \mathbf{y} \implies F(\mathbf{x}) \sim F(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then it gives rise to a unique transformation $f : \mathbb{T}^n \to \mathbb{T}^n$ satisfying $f \circ \pi = \pi \circ F$:

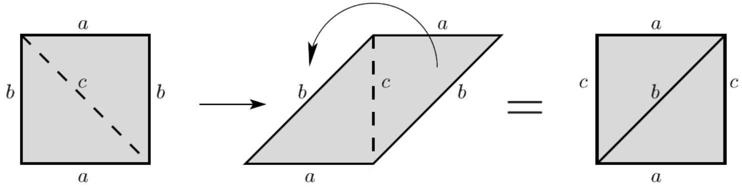
$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{F} & \mathbb{R}^n \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T}^n & \xrightarrow{f} & \mathbb{T}^n \end{array}$$

The map f is continuous (resp., smooth) if so is F.

Examples. • Translation (or rotation).
$F(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ is a constant vector.

• Toral endomorphism.
$F(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix with integer entries.

Example. $F(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.



Hyperbolic toral automorphisms

Suppose A is an $n \times n$ matrix with integer entries. Let L_A denote a toral endomorphism induced by the linear map $L(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$. The map L_A is a **toral automorphism** if it is invertible.

Proposition The following conditions are equivalent:

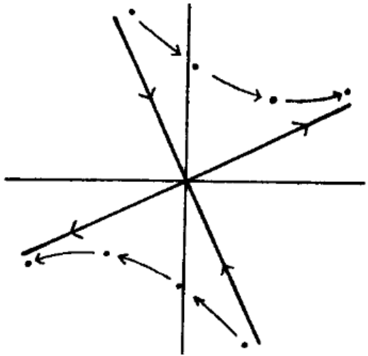
- L_A is a toral automorphism,
- A is invertible and A^{-1} has integer entries,
- $\det A = \pm 1$.

Definition. A toral automorphism L_A is **hyperbolic** if the matrix A has no eigenvalues of absolute value 1.

Theorem Every hyperbolic toral automorphism is chaotic.

Cat map

Example. L_A, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.



Stable and unstable subspaces project to dense curves on the torus.