MATH 614

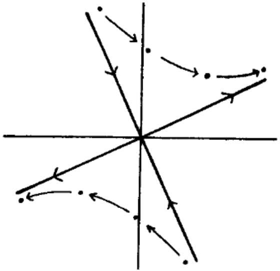
Dynamical Systems and Chaos

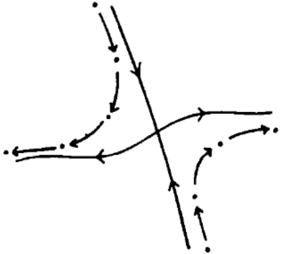
Lecture 17: Hyperbolic dynamics. Chain recurrence.

Saddle points

Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable map.

Definition. A periodic point p of period n of the map F is **hyperbolic** if the Jacobian matrix $DF^n(p)$ has no eigenvalues of absolute value 1 or 0. The hyperbolic periodic point p is a **sink** if every eigenvalue λ of $DF^n(p)$ satisfies $0 < |\lambda| < 1$, a **source** if every eigenvalue λ satisfies $|\lambda| > 1$, and a **saddle point** otherwise.



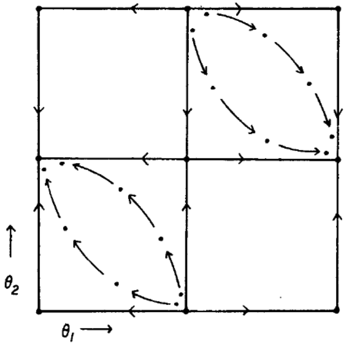


Example

In angular coordinates (θ_1, θ_2) on the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$,

$$F\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\theta_1 - \varepsilon \sin\theta_1\\ \theta_2 + \varepsilon \sin\theta_2\end{pmatrix}.$$

There are 4 fixed points: one source, one sink, and two saddles.

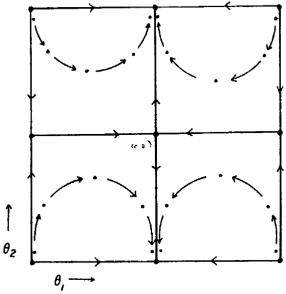


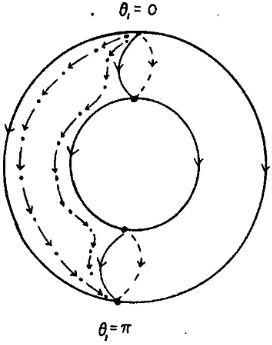
Example

In angular coordinates (θ_1, θ_2) on the torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$,

$$F\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\theta_1 + \varepsilon \sin\theta_1\\ \theta_2 + \varepsilon \sin\theta_2 \cos\theta_1\end{pmatrix}.$$

There are 4 fixed points: one source, one sink, and two saddles.





Chain recurrence

Suppose X is a metric space with a distance function d.
Let $F : X \to X$ be a continuous transformation.

Definition. A point $x \in X$ is **recurrent** for the map F if for any $\varepsilon > 0$ there is an integer $n > 0$ such that $d\big(F^n(x), x\big) < \varepsilon$. The point x is **chain recurrent** for F if, for any $\varepsilon > 0$, there are points $x_0 = x, x_1, x_2, \dots, x_k = x$ and positive integers $n_1, n_2, \dots, n_k$ such that $d\big(F^{n_i}(x_{i-1}), x_i\big) < \varepsilon$ for $1 \le i \le k$.

Examples.

- Any periodic point is recurrent.
- Any eventually periodic (but not periodic) point is not recurrent.
- If the orbit of x is dense in X, then x is recurrent unless x is an isolated point in X and not periodic for F.

- If $x \in W^s(p)$ for a periodic point p, then x is not recurrent unless $x = p$.

- If $X = S^1$ and F is a rotation then every point is recurrent (since either all points are periodic or all orbits are dense).

- If X is the torus $\mathbb{T}^n$ and F is a translation then every point is recurrent (since F preserves distances and volume).

- If $X = \Sigma_{\mathcal{A}}$ and $F = \sigma$ is the one-sided shift, then every point $\mathbf{s} \in X$ is chain recurrent. Indeed, let $\mathbf{s}^{(n)} = w_n w_n w_n \ldots,$ where w_n is the beginning of $\mathbf{s}$ of length n. Then $\sigma^n\big(\mathbf{s}^{(n)}\big) = \mathbf{s}^{(n)}$ and $\mathbf{s}^{(n)} \to \mathbf{s}$ as $n \to \infty$.

- If $X = \Sigma_{\mathcal{A}}$ and $F = \sigma$ is the one-sided shift, then not every point is recurrent. For example, $\mathbf{s} = (1000\ldots)$ is not recurrent.

- If $X = \Sigma_{\mathcal{A}}^{\pm}$ and $F = \sigma$ is the two-sided shift, then every point is chain recurrent but not every point is recurrent, e.g., $\mathbf{s} = (\ldots 000.1000\ldots)$.

Let $F : X \to X$ be a homeomorphism of a metric space X.

Definition. Suppose $x \in W^s(p) \cap W^u(q)$, where p and q are periodic points of F. Then x is called **heteroclinic** if $p \neq q$ and **homoclinic** if $p = q$.

- Any homoclinic point is chain recurrent.

Proposition 1 The set of all chain recurrent points is closed.

Proposition 2 Topological conjugacy preserves recurrence and chain recurrence.

- If $X = \mathbb{T}^2$ and F is a hyperbolic toral automorphism, then all points of X are chain recurrent (periodic points of F are dense and so are homoclinic points for the fixed point $[0, 0]$).

- If F is the logistic map $F(x) = \mu x(1 - x)$, $\mu > 4$, then all points of the invariant Cantor set Λ are chain recurrent.

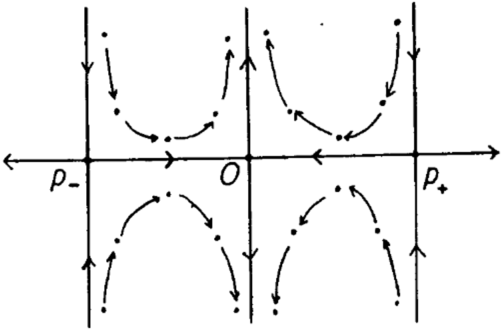
- If F is the horseshoe map, then all points of the invariant Cantor set Λ are chain recurrent.

Example

- $F(x, y) = (x_1, y_1)$, where $x_1 = \frac{1}{2}(x + x^3)$,

$y_1 = y \cdot \frac{2}{1 + 2x^2}$.

There are three fixed points: $p_+ = (1, 0)$, $p_- = (-1, 0)$ and $O = (0, 0)$. All three are saddle points.

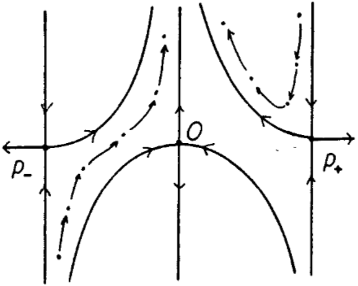


Example

- $F(x,y) = (x_1, y_1)$, where $x_1 = \frac{1}{2}(x + x^3)$,

$y_1 = y \cdot \frac{2}{1+2x^2} + \phi(|x|)$, where $\phi(t) > 0$ for $0 < t < 1$ and $\phi(t) = 0$ otherwise.

There are still three fixed points: $p_+ = (1,0)$, $p_- = (-1,0)$ and $O = (0,0)$. All three are still saddle points.



Morse-Smale diffeomorphisms

Definition. A diffeomorphism $F : X \to X$ is called **Morse-Smale** if

(i) it has only finitely many chain recurrent points,

(ii) every chain recurrent point is periodic,

(iii) every periodic point is hyperbolic,

(iv) all intersections of stable and unstable manifolds of saddle points of F are transversal.

Theorem (Palis) Any Morse-Smale diffeomorphism of a compact surface is C^1-structurally stable.