MATH 614

Dynamical Systems and Chaos

Lecture 20: Möbius transformations. Local holomorphic dynamics at fixed points.

Classification of periodic points

Let $U \subset \mathbb{C}$ be a domain and $F : U \to \mathbb{C}$ be a holomorphic function. Suppose that $F(z_0) = z_0$ for some $z_0 \in U$. The fixed point z_0 is called

- attracting if $|F'(z_0)| < 1$;
- repelling if $|F'(z_0)| > 1$;
- neutral if $|F'(z_0)| = 1$.

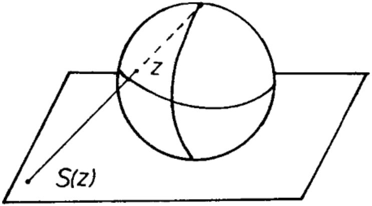
Now suppose that $F^n(z_1) = z_1$ for some $z_1 \in U$ and an integer $n \geq 1$. The periodic point z_1 is called

- attracting if $|(F^n)'(z_1)| < 1$;
- repelling if $|(F^n)'(z_1)| > 1$;
- neutral if $|(F^n)'(z_1)| = 1$.

The multiplier $(F^n)'(z_1)$ is the same for all points in the orbit of z_1. In particular, all these points are of the same type as z_1. Note that the multiplier is preserved under any holomorphic change of coordinates.

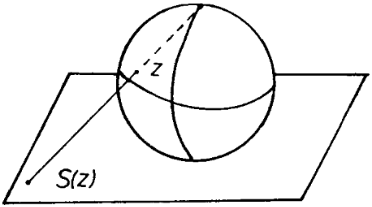
Stereographic projection

Suppose Σ is a sphere in $\mathbb{R}^3$ and Π is the tangent plane at some point $P_s \in \Sigma$. Let P_n be the point of Σ opposite to P_s. Then any straight line through P_n not parallel to Π intersects the plane Π and also intersects the sphere Σ at a point different from P_n.



This gives rise to a map $S : \Sigma \setminus \{P_n\} \to \Pi$, which is a homeomorphism. The map S is referred to as the **stereographic projection**. Note that S maps any circle on Σ onto a circle or a straight line in the plane Π.

The Riemann sphere



Introducing Cartesian coordinates on the plane Π with the origin at P_s, we can identify Π with the complex plane $\mathbb{C}$. The **extended complex plane** $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is $\mathbb{C}$ with one extra point "at infinity". We extend the stereographic projection S to a map $S : \Sigma \to \overline{\mathbb{C}}$ by letting $S(P_n) = \infty$. The topology on $\overline{\mathbb{C}}$ is defined so that S be a homeomorphism.

A holomorphic structure on $\mathbb{C}$ is extended to $\overline{\mathbb{C}}$ by requiring that the map $H : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by $H(z) = 1/z$ for $z \in \mathbb{C} \setminus \{0\}$, $H(0) = \infty$, and $H(\infty) = 0$ be holomorphic.

Möbius transformations

Definition. A **Möbius transformation** is a rational map of the form $T(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \neq 0$, regarded as a transformation of the Riemann sphere $\overline{\mathbb{C}}$.

Properties of the Möbius transformations:

- The Möbius transformations form a transformation group.
- Any Möbius transformation is a homeomorphism of $\overline{\mathbb{C}}$.
- Any Möbius transformation is holomorphic.
- Complex affine functions $T(z) = \alpha z + \beta$, $\alpha \neq 0$ are Möbius transformations that fix ∞.
- Complex linear functions $T(z) = \alpha z$, $\alpha \neq 0$ are Möbius transformations that fix 0 and ∞.
- The group of Möbius transformations is generated by linear functions $z \mapsto \alpha z$, translations $z \mapsto z + \beta$, and the map $z \mapsto 1/z$.

More properties of Möbius transformations

- Any Möbius transformation maps circles on the Riemann sphere (which are circles or straight lines in $\mathbb{C}$) onto other circles.
- For any triples z_1, z_2, z_3 and w_1, w_2, w_3 of distinct points on $\overline{\mathbb{C}}$ there exists a unique Möbius transformation T such that $T(z_i) = w_i, \; 1 \leq i \leq 3$.
- Any Möbius transformation different from the identity has one or two fixed points.
- Any Möbius transformation is conjugate (by another Möbius transformation) to a translation or a linear map.
- If a Möbius transformation has only one fixed point, then this fixed point is neutral.
- If a Möbius transformation has two fixed points, then either both are neutral, or one is attracting while the other is repelling.

Hyperbolic fixed points

Theorem 1 Suppose z_0 is an attracting fixed point for a holomorphic function F. Then there exist $\delta > 0$ and $0 < \mu < 1$ such that

$$|F(z) - z_0| \leq \mu |z - z_0|$$

for any $z \in D = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. In particular, $\lim_{n\to\infty} F^n(z) = z_0$ for all $z \in D$.

Theorem 2 Suppose z_0 is a repelling fixed point for a holomorphic function F. Then there exist $\delta > 0$ and $M > 1$ such that

$$|F(z) - z_0| \geq M |z - z_0|$$

for all $z \in D = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. In particular, for any $z \in D \setminus \{z_0\}$ there is an integer $n \geq 1$ such that $F^n(z) \notin D$.

Hyperbolic fixed points

Theorem 3 Let F be a holomorphic function at 0 such that $F(0) = 0$ and $F'(0) = \lambda$, where $0 < |\lambda| < 1$. Then there is a neighborhood U of 0 and a holomorphic map $h : U \to \mathbb{C}$ such that $F \circ h = h \circ L$ in U, where $L(z) = \lambda z$.

The map h is represented as $h(z) = z + \sum_{i=2}^{\infty} c_i z^i$.

Theorem 4 Let F be a holomorphic function at 0 such that $F(0) = 0$ and $F'(0) = \lambda$, where $|\lambda| > 1$. Then there is a neighborhood U of 0 and a holomorphic map $h : U \to \mathbb{C}$ such that $F \circ h = h \circ L$ in $L^{-1}(U)$, where $L(z) = \lambda z$.

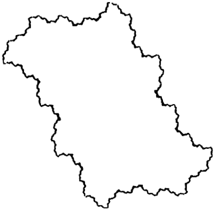
Basin of attraction

Suppose z_0 is an attracting fixed point of a holomorphic map F. Then the set D_{z_0} of all points z satisfying $F^n(z) \to z_0$ as $n \to \infty$ is open. This set is called the **basin of attraction** for z_0. The connected component of D_{z_0} containing z_0 is called the **immediate basin of attraction** for z_0.

Examples. • $F(z) = z^2$.

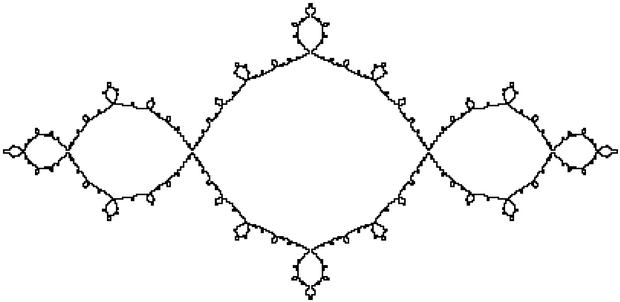
0 is an attracting fixed point. The basin of attraction for 0 is the unit disc $\{z \in \mathbb{C} : |z| < 1\}$, it is connected.

• $F(z) = z^2 + i/2$.



Example

- $F(z) = z^2 - 1$.



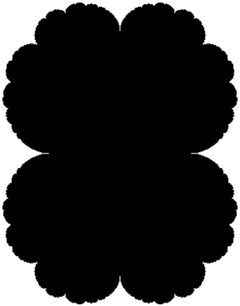
0 and -1 form a periodic orbit, which is attracting. The basin of attraction for each of these points has infinitely many connected components. Each component contains a unique eventually periodic point.

Also, there are two fixed points, which are repelling.

Neutral fixed points

Example. • $F(z) = z + z^2$.

The map has a fixed point at 0, which is neutral: $F'(0) = 1$. The set D_0 of all points z satisfying $F^n(z) \to 0$ as $n \to \infty$ is open and connected.



The fixed point 0 is one of the cusp points at the boundary of D_0. The others correspond to eventually fixed points.